MS323 Block I Unit 1

The Open University

Mathematics/Science:
An Interfaculty Third Level Course

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INTRODUCTION TO NON–LINEAR DYNAMICS

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The Open University, Walton Hall, Milton Keynes, MK7 6AA.

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2.1

Contents

Unit 1 Preliminaries 5

1.1 Introduction 5
  1.1.1 Introduction to the course 5
  1.1.2 Introduction to this unit 7

1.2 Calculus 7
  1.2.1 Some exercises in basics 7
  1.2.2 Functions of several variables 9
  1.2.3 The gradient 12
  1.2.4 Taylor polynomials 14
  1.2.5 The chain rule 15
  1.2.6 Extending the chain rule 17

1.3 Differential equations 19
  1.3.1 Analytical methods 19
  1.3.2 Graphical methods 22

1.4 Matrices 24
  1.4.1 Vector notation 24
  1.4.2 Matrix algebra 24
  1.4.3 Determinants 26
  1.4.4 Eigenvalues and eigenvectors 28
  1.4.5 Complex eigenvalues 32
  1.4.6 Jacobian matrices 34

1.5 Systems of differential equations 42

1.6 Newtonian mechanics 45
  1.6.1 Motion of a particle along a straight line 45
  1.6.2 Energy 47
  1.6.3 Momentum 48
  1.6.4 Circular motion 48
  1.6.5 Angular momentum 51
  1.6.6 Dimensions 52

Solutions to the Exercises in Unit 1 55

UNIT 1

PRELIMINARIES

1.1 Introduction

1.1.1 Introduction to the course

Dynamics is concerned with how things change in time. Newtonian mechanics provides one class of examples which you will study. In fact, in the context of physical systems subject to forces, the term 'dynamics' is used almost interchangeably with 'mechanics'. In this course, although mechanical systems are important, they are by no means the only kind of system studied; the course is about systems undergoing change, however caused, and the focus is on the analysis of the changes themselves rather than on their causes.

The systems in question may not move about in space, as a particle on a spring does. For example, the feature of interest in a biological system could be the population of some species of animal with a limited supply of food. The animals move about, it is true; but this motion is not of any interest – it is the motion of the population size that will concern us. It may sound odd to say that the population size moves. However, its value at any given time may be recorded graphically as a point, and as time goes on and the population varies this representative point moves. Any system changing in time may be represented in this way, by choosing parameters of interest to specify the state of the system at any one time, taking these parameters as coordinates in a space of suitable dimension, and picturing the change of the system as the motion of the representative point.

The space in which the time development of a system takes place, in this graphical representation, is called its phase space or state space. The dimension of the phase space – the number of coordinates required to specify a point in it – is equal to the number of parameters required to specify the state of the system. A very important aspect of the specification of the state of a system is that the time development of the system, initially in a given state, must be uniquely determined. For example, in the case of a Newtonian mechanical system consisting of a single particle moving along a line, the phase space is two-dimensional, not one-dimensional as one might expect, because knowledge of the particle's initial velocity, as well as its initial position, is necessary in order to determine its subsequent motion. Thus the dimension of the phase space of a system is not always completely obvious.

This course deals with systems whose phase spaces are at most two-dimensional. You may perhaps imagine that systems with such low-dimensional phase spaces are simple to analyse and not particularly interesting. This is far from the truth; in fact, in order to keep the subject matter within the compass of a half-credit course, attention must be restricted for the most part to just one particular class of dynamical systems. The systems with which the course is mostly concerned are called Hamiltonian systems, after Sir William Rowan Hamilton (1805–65). This class includes many dynamical systems of practical interest, such as conservative Newtonian mechanical systems (those in which the law of conservation of energy holds). We explain what a Hamiltonian system is in *Unit 4*.

There is another classification of dynamical systems. It depends on consideration of how time enters the reckoning. Contrast the time development of a population of some species of animal, say, which has an annual breeding cycle, with the time development of a mechanical system such as a particle on a perfect spring. In the first example it would be enough to record the population size once each year, say at the end of the breeding season. One would expect the population size in the current year to be determined by a formula involving the population size in the previous year, and perhaps earlier years; in any event, by a recurrence relation. For the particle on the spring, on the other hand, every instant counts, and the motion is determined by a differential equation. A system whose state is specified at discrete instants of time, and whose time development is given by a recurrence relation (or relations), is said to be a discrete system; a species with an intermittent breeding pattern, whose population is of interest, is an example. A system whose state may change continuously, and whose time development is determined by a differential equation (or equations), is said to be a continuous system; any Newtonian mechanical system is an example. This may appear to contradict what we have just been saying, as populations are often modelled as continuous systems; but we trust that the distinction is clear. This course is concerned mostly with continuous systems. However, some of the most exciting recent developments in the study of dynamical systems, for example those concerned with chaotic behaviour, are most simply described for discrete systems, and some space is devoted to them at the end of the course.

One rather simplistic definition of the subject matter of dynamics is that it is the study of differential equations (for continuous systems) or of recurrence relations (for discrete systems). This should indeed invoke an appropriate state of mind in which to begin the course. But it may suggest that the course is concerned with developing more and more sophisticated methods of solving differential equations and recurrence relations, whether analytically or numerically. In fact, we shall have little to say about methods of solution, and most of what we do say on this subject is revision of material from *MST204 Mathematical Models and Methods* and is contained in this unit. The concern is mostly with qualitative rather than quantitative aspects of the solutions of differential equations and recurrence relations. Thus, in the case of a particle on a spring, we shall be concerned not so much with questions like 'how long does it take for the particle to execute one complete oscillation', but more with the very fact that the motion is oscillatory, and what it is about the system, from the mathematical point of view, that makes it so.

An important aspect of this qualitative approach to the study of dynamical systems is that it combines analytical and geometrical ideas. In many cases, the basic tactic is to study a system by representing it as a direction field, or flow, in phase space, and by sketching solutions as trajectories of this field. Features of interest which will be covered are fixed points and closed trajectories (closed in the sense that a circle is closed, as opposed to a line), and the behaviour of the system near such features will also be of prime importance. In the real-life motion of the system, a fixed point of the phase space flow corresponds to a state in which the system remains stationary, in other words, an equilibrium state; and a closed trajectory corresponds to a periodic motion. The study of a system's behaviour near an equilibrium state or periodic motion amounts to the study of the stability of the state or motion. So the stability of equilibrium and oscillatory motion will be an important part of the course.

The title of the course, 'Introduction to Non-linear Dynamics', may suggest that linear dynamics gets short shrift. Actually the title means 'not necessarily linear' dynamics. Here we are using the word 'linear' in much the same sense that it is used in such a phrase as 'linear differential equation'. But despite the fact that we have thrown open the door to all sorts of dynamical systems as legitimate objects of study in this course, linear systems still have a very important role to play.

This is basically because truly non-linear systems are much more complicated to analyse than are linear ones, and a key tool in studying non-linear systems is to approximate them by linear ones. So a fair amount of effort, especially in the first few units, is devoted to the study of linear systems.

1.1.2 Introduction to this unit

The main purpose of this unit is to review the topics which will be needed from *MST204* (which is the recommended prerequisite for the present course) and where necessary extend them to match their use in *MS323*. This is, in short, a bridging unit; it covers a lot of material, most of which you should be familiar with, and is longer than later units so we allow you more time to study it.

The topics covered are those you would expect from reading the introduction above: differential equations and systems of differential equations; calculus, including calculus of functions of several variables; matrices, which are needed for solving systems of differential equations; Newtonian mechanics of particles in one dimension. It is a mixed bag of subjects, and there is no very obvious logical order in which to present them.

We have kept this unit separate from the others, which are bound together in blocks, because we imagine that it may be helpful to be able to refer to this unit while studying the others, especially those in the first block. However, it will probably be most efficient for you to work through this unit once before starting on the new material, even if you do find that you need to refer again to parts of it when you actually come to apply them.

1.2 Calculus

1.2.1 Some exercises in basics

To brush up your calculus of functions of one variable, you may like to try the following sequence of exercises, which deal with the hyperbolic cosine and sine functions, which are defined in terms of the exponential function by

$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^{x} - e^{-x}).$$

These functions should be familiar to you from *MST204*, but are probably less familiar than, say, the trigonometrical functions such as $\cos x$ and $\sin x$. They are used from time to time in this course, so the exercises will have a dual role, reminding you of their properties as well as of the fundamental processes of calculus.

Unit 18, Section 3

Incidentally, we use a notation for integrals which is slightly different from the one you will be familiar with from *MST204*. We think that it is preferable to think of $\int dx$ as a single indivisible symbol, which stands for the instruction 'integrate with respect to x'; and we therefore write (in Exercise 1.9(i)) $\int dx \sinh x$ rather than $\int \sinh x \, dx$. This is consistent with the usual notation for differentiation, in which $/dx$ is considered as a single indivisible symbol meaning 'differentiate with respect to x'. However, some departures from the strict application of this rule will creep in, as reference to Exercise 1.9(iii) and later sections will show!

Another difference in notation between this course and *MST204* is that we shall use ln for the natural logarithm function, log to base e, rather than $\log_e$. We think ln is clearer, because it prevents confusion with logs to other bases, as well as being more economical of writing.

Exercise 1.1

Using the above definitions, show that

$$\frac{d}{dx}\cosh x = \sinh x \qquad \text{and} \qquad \frac{d}{dx}\sinh x = \cosh x.$$

Exercise 1.2

By calculating its derivative, show that $\cosh^2 x - \sinh^2 x$ is constant, and show that it is in fact equal to 1.

Exercise 1.3

We define also

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x},$$
$$\operatorname{cosech} x = \frac{1}{\sinh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Find the derivatives of these functions.

Exercise 1.4

Show that $\sinh x$ is an increasing function and is odd, while $\cosh x$ has a single minimum and is even. Sketch the graphs of the two functions.

Exercise 1.5

Sketch the graph of $\tanh x$.

Exercise 1.6

By considering the graph of $y = \sinh x$, show that the inverse function $x(y)$ is a single-valued function of y. Further, show that x and y satisfy the equation

$$e^{2x} - 2ye^{x} - 1 = 0,$$

and deduce that

$$x(y) = \ln(y + \sqrt{y^2+1}).$$

Similarly, show that the inverse function of the equation $y = \cosh x$ is a two-valued function, and find explicit expressions for $x(y)$.

The function $\ln(y + \sqrt{y^2+1})$ is thus the inverse of the sinh function and is denoted by $\sinh^{-1} y$, or sometimes arcsinh y. Also, $\ln(y + \sqrt{y^2-1})$ is denoted by $\cosh^{-1} y$ (so that $\cosh^{-1} y \geq 0$). In this course we shall use similar notation for the inverse trigonometric functions, as follows:

These notations replace arctan and so on used in *MST204*.

$$\begin{array}{lll} y = \sin x, & x = \sin^{-1} y, & -\pi/2 \leq x \leq \pi/2, \\ y = \cos x, & x = \cos^{-1} y, & 0 \leq x \leq \pi, \\ y = \tan x, & x = \tan^{-1} y, & -\pi/2 < x < \pi/2. \end{array}$$

Exercise 1.7

Show that for any constants A and B, $y = A\cosh x + B\sinh x$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - y = 0.$$

Show that $\cosh(x+a)$, where a is a constant, also satisfies this differential equation, and deduce the addition formula

$$\cosh(x+a) = \cosh x \cosh a + \sinh x \sinh a.$$

Find a similar formula for $\sinh(x+a)$. Derive formulae for $\cosh 2x$ and $\sinh 2x$ from your results.

Exercise 1.8

Show that the Taylor series for the two functions are

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots,$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots.$$

Exercise 1.9

Evaluate the following integrals.

(i) $\int dx \sinh x$

(ii) $\int dx \cosh x$

(iii) $\int \frac{dx}{\sqrt{x^2 - a^2}} \quad (|x| > a > 0)$

(iv) $\int dx \sqrt{x^2 - a^2} \quad (x > a > 0)$

(v) $\int \frac{dx}{\sqrt{x^2 + a^2}} \quad (a > 0)$

(vi) $\int dx \tanh x$

[Hint: In some of these integrals, the relations derived in Exercises 1.1, 1.2 and 1.7 are useful.]

1.2.2 Functions of several variables

Since much of the course will be concerned with systems with two-dimensional phase spaces, functions of two variables will be encountered repeatedly.

MST204 Unit 25

You will recall that the basic process of differentiation of a function of several variables is the process of partial differentiation, which amounts to differentiation with respect to one of the variables, regarding the others temporarily as constants. The usual rules of differentiation apply, with one additional important one which comes into play with second- and higher-order derivatives: the mixed derivative rule. This says that, for any well-behaved function, the order of repeated partial differentiation with respect to different variables does not matter. The meaning of the term 'well-behaved' need not detain you; it is enough to say that any function which can be expressed as an algebraic combination of polynomials and the standard trigonometrical and exponential functions, which can be differentiated at all, will be well-behaved; and all functions you are likely to meet in this course will be well-behaved. Thus in the case of a function f of two variables x and y,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}. \qquad (1.1)$$

Example 1.1

Calculate the first- and second-order partial derivatives of the function

$$f(x, y) = x^3 + 2x^2 y + xy + y.$$

Solution

The partial derivatives are as follows.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3x^2 + 4xy + y, \\
\frac{\partial f}{\partial y} &= 2x^2 + x + 1, \\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x + 4y, \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 4x + 1, \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 4x + 1, \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0.
\end{aligned}$$

This is all quite straightforward. Note that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

as promised by the mixed derivative rule. In practice it is never necessary to calculate both mixed derivatives, for a well-behaved function, because of that rule; it is done here merely to make it clear what the rule means.

If you were to continue to third-order derivatives, in order to calculate $\partial^3 f/\partial x \partial y^2$ you could either partially differentiate $\partial^2 f/\partial y^2$ with respect to x or partially differentiate $\partial^2 f/\partial x \partial y$ with respect to y; the answer would be the same in either case (as it happens, 0 for this example). ☐

There is more to the mixed derivative rule than calculational convenience: it can often be used to extract more information out of a situation involving functions of several variables than might at first sight appear possible. Consider the following example.

Example 1.2

What functions f of two variables can satisfy the following conditions:

$$\begin{aligned}
&\frac{\partial f}{\partial x} \text{ is a function of } y \text{ alone,} \\
&\frac{\partial f}{\partial y} \text{ is a function of } x \text{ alone?}
\end{aligned}$$

Solution

On the face of it, there is not much one can say about such a function. However, set

$$\frac{\partial f}{\partial x} = g(y), \qquad \frac{\partial f}{\partial y} = h(x).$$

Assuming that f is well-behaved,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{dh}{dx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{dg}{dy}.$$

But dh/dx is a function of x and dg/dy is a function of y, and the two cannot be equal unless each is constant. If

$$\frac{dh}{dx} = \frac{dg}{dy} = k,$$

then on integrating,

$$h(x) = kx + m, \qquad g(y) = ky + l,$$

where m and l are also constants. ☐

This solution is completed in the exercise below.

Exercise 1.10

Show that $f(x, y) = g(y)x + a(y)$ for some unknown function $a(y)$, and deduce that $da/dy = m$. Conclude that

$$f(x, y) = kxy + lx + my + n,$$

where k, l, m and n are all constants.

Exercise 1.11

Show that there can be no (well-behaved) function $h(x, y)$ such that

$$\frac{\partial h}{\partial x} = x^2 + y, \qquad \frac{\partial h}{\partial y} = -x + y^2.$$

One situation in which partial differentiation is sometimes unexpectedly useful is in the evaluation of integrals, as the following example illustrates.

Example 1.3

Evaluate the function $f(y)$ defined by the integral

$$f(y) = \int_0^\infty dx\, xe^{-yx}, \qquad y > 0.$$

Solution

This integral may be evaluated straightforwardly using integration by parts:

$$\begin{aligned}\int_0^\infty dx\, xe^{-yx} &= \int_0^\infty dx \left\{ \frac{d}{dx}\left(-\frac{1}{y}xe^{-yx}\right) + \frac{1}{y}e^{-yx} \right\} \\ &= \left[-\frac{1}{y}xe^{-yx}\right]_0^\infty + \frac{1}{y}\int_0^\infty dx\, e^{-yx} \\ &= -\frac{1}{y^2}\left[e^{-yx}\right]_0^\infty \\ &= \frac{1}{y^2}.\end{aligned}$$

Note that since $y > 0$, $e^{-yx} \to 0$ as $x \to \infty$.

There is, however, a quicker way of evaluating the integral, which involves noting that the integrand is the partial derivative with respect to y of $-e^{-yx}$. Thus

$$\begin{aligned}\int_0^\infty dx\, xe^{-yx} &= \int_0^\infty dx \frac{\partial}{\partial y}\left(-e^{-yx}\right) \\ &= -\frac{d}{dy}\left(\int_0^\infty dx\, e^{-yx}\right).\end{aligned}$$

The last step is justified because the integration is with respect to x, not y, which is a constant as far as the integration is concerned. It does not matter in which order one carries out the operations of integration (with respect to x) and partial differentiation (with respect to y), just as the order of partial differentiation is irrelevant. Now the new integral is very easy to evaluate:

$$\int_0^\infty dx\, e^{-yx} = \left[-\frac{1}{y}e^{-yx}\right]_0^\infty = \frac{1}{y},$$

and so

$$f(y) = -\frac{d}{dy}\int_0^\infty dx\, e^{-yx} = \frac{1}{y^2},$$

which agrees with the first answer, as it should. ☐

The principle used in solving this problem by the second method applies to any integral whose integrand depends upon a parameter, say y, provided that the integrand is a well-behaved function. The integral may be regarded as defining a function of y, whose derivative may be calculated by partially differentiating the integrand with respect to y before carrying out the integration, thereby reversing the order of integration and differentiation.

Thus, in general, if

$$f(y) = \int_a^b dx\, F(x, y)$$

then

$$\frac{df}{dy} = \int_a^b dx\, \frac{\partial F}{\partial y},$$

provided that the integrals exist.

A more complicated variant of this result applies when one or both of the limits of the integral also depend upon y. If

$$f(y) = \int_{a(y)}^{b(y)} dx\, F(x, y)$$

then

$$\frac{df}{dy} = b'(y)F(b(y), y) - a'(y)F(a(y), y) + \int_{a(y)}^{b(y)} dx\, \frac{\partial F}{\partial y}. \tag{1.2}$$

Exercise 1.12

Find an explicit expression for the function $f(y)$ defined by the integral

$$f(y) = \int_0^\pi dx\, x^2 \sin xy.$$

[Hint: In this case you will need to differentiate twice.]

1.2.3 The gradient

One important use of partial differentiation is in the calculation of the gradient of a function of two variables, or scalar field as such a function is often called. The gradient grad h of a scalar field $h(x, y)$ is the vector field whose components are the first-order partial derivatives of h:

MST204 Unit 26, Section 2

$$\operatorname{grad} h = \frac{\partial h}{\partial x}\mathbf{i} + \frac{\partial h}{\partial y}\mathbf{j}, \tag{1.3}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the positive x- and y-axes.

The result is a formula for calculating a vector at each point in the (x, y)-plane: hence the term 'vector field' (in contrast to the function h itself, which is a formula for calculating a scalar at each point in the (x, y)-plane). The direction of grad h at a point is perpendicular to the contour curve, or curve of constant h, passing through the point. The rate of change of h in a given direction at the point is $\mathbf{e} \cdot \operatorname{grad} h$, where $\mathbf{e}$ is a unit vector in the given direction; h changes fastest when $\mathbf{e}$ points in the direction of grad h.

Example 1.4

Let

$$h(x, y) = x^2 + y^2.$$

Describe the contour curves of h, find grad h, and confirm that its direction is indeed perpendicular to the contours, for $(x, y) \neq (0, 0)$.

Solution

One may write $x^2 + y^2 = r^2$, where r is the usual radial coordinate in polar coordinates. The contour curves of h are the curves on which h is constant, so the curves on which r is constant; these are just the circles centred at the origin.

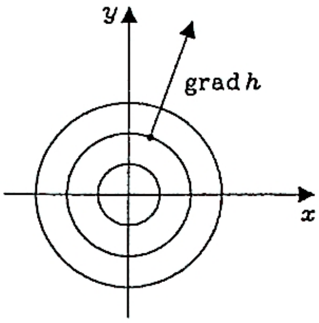

Figure 1.1

From

$$\operatorname{grad} h = 2x\,\mathbf{i} + 2y\,\mathbf{j},$$

it can be seen that at each point $(x,y) \neq (0,0)$, grad h is just a constant multiple of the position vector $x\,\mathbf{i} + y\,\mathbf{j}$ of the point: it is thus a multiple of the radius vector of the circle through the point, and this is perpendicular to the circle. At the origin the gradient vanishes, and therefore cannot be perpendicular to anything. (The origin is, of course, the minimum of the function h.) □

Exercise 1.13

Find grad h if $h(x,y) = ax^2 + 2bxy + cy^2$, where a, b and c are constants.

Exercise 1.14

Show that the vector field $u(x,y)\,\mathbf{i} + v(x,y)\,\mathbf{j}$ cannot be the gradient of any function unless

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}.$$

Exercise 1.15

Find a unit vector perpendicular to the line $3x - 4y = 1$.

In Exercise 1.14 you showed that if $u(x,y)\,\mathbf{i} + v(x,y)\,\mathbf{j}$ is a gradient, then $\partial v/\partial x = \partial u/\partial y$. This result has a converse: if the condition on the partial derivatives is satisfied, then the vector field is a gradient. You may recall from *MST204* the use of the curl as a test of whether or not a vector field is a gradient, or in other words whether or not it is conservative; for vector fields in two dimensions, the curl test reduces to the single condition above. It is, in fact, quite easy to give a formula for a scalar field $F(x,y)$ which has gradient $u(x,y)\,\mathbf{i} + v(x,y)\,\mathbf{j}$ in terms of integrals:

Unit 26, Section 4

$$F(x,y) = \int_0^x d\xi\, u(\xi,0) + \int_0^y d\eta\, v(x,\eta) + F(0,0). \tag{1.4}$$

To show that grad F is the given vector field, assuming of course that $\partial v/\partial x = \partial u/\partial y$, proceed as follows (using Equation 1.2):

$$\begin{aligned}\frac{\partial F}{\partial x}(x,y) &= u(x,0) + \int_0^y d\eta\, \frac{\partial v}{\partial x}(x,\eta) \\ &= u(x,0) + \int_0^y d\eta\, \frac{\partial u}{\partial \eta}(x,\eta) \\ &= u(x,0) + u(x,y) - u(x,0) = u(x,y).\end{aligned}$$

This calculation has used the fact that differentiation is the reverse of integration to deal with the integral involving u. The same result leads immediately to the required formula for the other partial derivative:

$$\frac{\partial F}{\partial y}(x,y) = v(x,y).$$

1.2.4 Taylor polynomials

The Taylor polynomials of a function provide a way of approximating the function near a point by means of polynomials whose coefficients are given in terms of the derivatives of the function at that point. The method of calculating Taylor polynomials for a function of one variable was exemplified in Exercise 1.8 above. In dealing with a function of several variables, the derivatives appearing in the Taylor polynomials are, of course, partial derivatives. If $f(x, y)$ is a well-behaved function of two variables, then its first-order Taylor polynomial at the point (a, b) is

MST204 Unit 25, Section 2

$$f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b), \tag{1.5}$$

which is a function linear in x and y, that is to say, a polynomial of order 1 (unless both partial derivatives happen to vanish at (a, b), in which case it is of order 0).

The first-order Taylor polynomial corresponds geometrically to the tangent plane to the surface $z = f(x, y)$ in three-dimensional space, defined by the function f, at the point $(a, b, f(a, b))$. The equation of this tangent plane is

$$z = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b). \tag{1.6}$$

When we use the first-order Taylor polynomial to approximate a function, we shall often write, for example,

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) + O\big(|\mathbf{r}-\mathbf{a}|^2\big), \tag{1.7}$$

where $\mathbf{r} = (x, y)$ and $\mathbf{a} = (a, b)$, and the expression $O\big(|\mathbf{r}-\mathbf{a}|^2\big)$ indicates that the terms which are being ignored are of order two and higher in $(x - a)$ and $(y - b)$ (O stands for 'order').

Exercise 1.16

Let $\mathbf{e} = \xi\mathbf{i} + \eta\mathbf{j}$ be a unit vector. By using the first-order Taylor approximation to $F(t) = f(a + t\xi, b + t\eta)$, show that

$$\frac{dF}{dt}(0) = \mathbf{e}\cdot\operatorname{grad} f,$$

where the gradient is evaluated at (a, b), and so confirm the result given above (page 12) about the rate of change of a scalar field in the direction determined by a unit vector.

The Taylor polynomial of second order at (a, b) for $f(x, y)$ is given by

$$\begin{aligned} &f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \tfrac{1}{2}(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + (x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}(a,b) + \tfrac{1}{2}(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b); \end{aligned} \tag{1.8}$$

this result is proved in the next exercise.

Exercise 1.17

By treating y as constant and using the Taylor approximation of order two for a function of one variable, show that

$$f(x,y) \simeq f(a,y) + (x-a)\frac{\partial f}{\partial x}(a,y) + \tfrac{1}{2}(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,y).$$

By using the Taylor approximation again, deduce that

$$\begin{aligned} f(x,y) \simeq{}& f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &+ \tfrac{1}{2}(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + (x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}(a,b) + \tfrac{1}{2}(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b). \end{aligned}$$

1.2.5 The chain rule

In *MST204* it is shown that if h is a function of two variables x and y, and these two variables are in turn functions of a single variable t, then

Unit 25, Section 2

$$\frac{dh}{dt}=\frac{\partial h}{\partial x}\frac{dx}{dt}+\frac{\partial h}{\partial y}\frac{dy}{dt}. \tag{1.9}$$

An example should remind you of what this means.

Example 1.5

Let

$$h(x,y)=x^3+y^2,$$
$$x(t)=1+t, \qquad y(t)=(1-t)^2.$$

Evaluate dh/dt directly and by using the chain rule.

Solution

To evaluate dh/dt directly, first express h as a function of t, by substituting for x and y in terms of t:

$$\begin{aligned} h(t) &= x(t)^3+y(t)^2 \\ &= (1+t)^3+(1-t)^4 \\ &= (1+3t+3t^2+t^3)+(1-4t+6t^2-4t^3+t^4) \\ &= 2-t+9t^2-3t^3+t^4. \end{aligned}$$

Thus

$$\frac{dh}{dt}=-1+18t-9t^2+4t^3.$$

To use the chain rule, on the other hand, first calculate the partial derivatives of h with respect to x and y, and the derivatives of x and y with respect to t:

$$\frac{\partial h}{\partial x}=3x^2, \qquad \frac{\partial h}{\partial y}=2y,$$
$$\frac{dx}{dt}=1, \qquad \frac{dy}{dt}=-2(1-t).$$

To find dh/dt, combine these expressions according to the formula 1.9 given by the chain rule; but, remembering that the final result is to be given in terms of t alone, it is worth substituting for x and y in terms of t in the partial derivatives before taking the final step:

$$\frac{\partial h}{\partial x}(x(t),y(t))=3x(t)^2=3(1+t)^2,$$
$$\frac{\partial h}{\partial y}(x(t),y(t))=2y(t)=2(1-t)^2.$$

So

$$\begin{aligned} \frac{dh}{dt} &= 3(1+t)^2\times 1+2(1-t)^2\times(-2(1-t)) \\ &= 3(1+2t+t^2)-4(1-3t+3t^2-t^3) \\ &= -1+18t-9t^2+4t^3, \end{aligned}$$

as before. ☐

Exercise 1.18

Let

$$h(x,y)=x^2+xy-y^2,$$
$$x(t)=2t, \qquad y(t)=1-3t.$$

Evaluate dh/dt directly and by using the chain rule.

The chain rule can be thought of as providing a procedure for calculating the derivative of the composite function $h(x(t), y(t))$, which is an alternative to the direct method of substituting for x and y in terms of t and differentiating. The alternative is to calculate the partial derivatives of h first, express the results in terms of t, multiply each partial derivative by the derivative of x or y as appropriate, and add the results, which is sometimes easier. The chain rule is often useful in obtaining theoretical results.

The chain rule can be interpreted geometrically, as follows. The functions $x(t)$ and $y(t)$ may be thought of as specifying a curve in the (x, y)-plane in terms of a parameter t (it may be helpful to think of t as the time, though it is not necessary to do so). The position vector $\mathbf{r}(t)$ of a point on this curve is given by

MST204 Unit 14, Section 4

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}. \tag{1.10}$$

Then

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \tag{1.11}$$

is a tangent vector to the curve at the point in question (and, if t is the time, it is the velocity vector of a point moving along the curve).

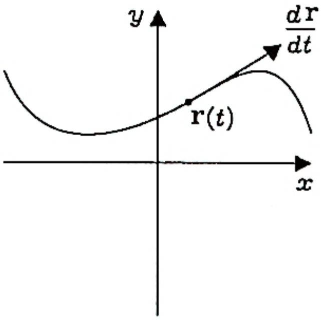

Figure 1.2

The function h is a scalar field on the plane. The function $h(x(t), y(t))$ is the restriction of this scalar field to the curve, and so dh/dt measures the rate of change of h as one moves along the curve. The partial derivatives of h are the components of its gradient:

$$\operatorname{grad} h = \frac{\partial h}{\partial x}\mathbf{i} + \frac{\partial h}{\partial y}\mathbf{j}. \tag{1.12}$$

The chain rule therefore says that

$$\frac{dh}{dt} = \operatorname{grad} h \cdot \frac{d\mathbf{r}}{dt}; \tag{1.13}$$

in words, that the rate of change of the scalar field h along the curve can be computed by taking the dot product of the gradient of h with the tangent vector to the curve.

It is worth noting in particular that the chain rule shows that, for a given function h, the derivative dh/dt depends linearly on the components of the tangent vector to the curve (in other words, it is a linear combination of those components). Also, for a given curve, dh/dt depends linearly on grad h. First derivatives are concerned with taking linear approximations – this is an extremely important principle.

1.2.6 Extending the chain rule

The chain rule presented in the previous subsection is but one example of a whole class of results about the derivatives of functions formed by composition (that is to say, by substitution for one or more variables in terms of others). The simplest of these is the chain rule for functions of one variable: if g is a function of x, and x is a function of t, then

$$\frac{dg}{dt} = \frac{dg}{dx}\frac{dx}{dt}. \tag{1.14}$$

It is pointed out in *MST204* that the chain rule generalises to a function h of three variables, x, y and z, as follows:

Unit 25

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} + \frac{\partial h}{\partial z}\frac{dz}{dt}. \tag{1.15}$$

Here, imagine that x, y and z are all given as functions of t, the term on the left being

$$\frac{d}{dt}(h(x(t), y(t), z(t))),$$

the derivative of the function of t obtained by substituting for x, y and z in terms of t in the formula for h.

Exercise 1.19

Interpret the chain rule for functions of three variables in terms of gradients and tangent vectors.

Exercise 1.20

Suppose that h is a function of three variables, x, y and z, and that the first two, x and y, are both given as functions of the third, z. Find a formula for dh/dz in terms of the partial derivatives of h with respect to x, y and z, and the ordinary derivatives of x and y with respect to z.

Exercise 1.21

Suppose that h is a function of n variables, $x_1, x_2, \ldots, x_n$, and that each of these variables is a function of the single variable t. Suggest a generalisation of the chain rule which applies to this situation.

Consider next a function g of one variable, say z, which is itself a function of two variables, x and y as before. From these may be formed a new function of two variables $g(z(x, y))$ by substituting for the variable z in terms of x and y in the formula for g. The chain rule extends to cover this situation, and gives a formula for each of the partial derivatives of the composite function in terms of the ordinary derivative of g and the partial derivatives of z. The correct formula is easily deduced from the simplest case of the chain rule. The process of partial differentiation of $g(z(x, y))$ with respect to x (for example) may be regarded as being equivalent to ordinary differentiation with y being treated as a constant. The chain rule for functions of one variable gives

$$\frac{\partial g}{\partial x} = \frac{dg}{dz}\frac{\partial z}{\partial x}, \tag{1.16}$$

where partial derivatives are used where it is necessary to express the fact that the differentiation is with respect to x keeping y constant. Similarly,

$$\frac{\partial g}{\partial y} = \frac{dg}{dz}\frac{\partial z}{\partial y} \tag{1.17}$$

for the partial derivative with respect to y.

Exercise 1.22

Let f be a function of a single variable. Define a scalar field h by

$$h(x,y) = f(r), \qquad \text{where} \quad r^2 = x^2 + y^2.$$

Show that

$$\operatorname{grad} h = \frac{df}{dr}\mathbf{e}_r,$$

where $\mathbf{e}_r = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j}$, which is the unit vector in the radial direction.

The next twist of the screw is to consider a function h of two variables, u and v, each of which is a function of two variables, x and y. Form the composite function $h(u(x,y), v(x,y))$, and ask for a formula for the partial derivatives of this composite function (with respect to x and y) in terms of the partial derivatives of h with respect to u and v, and the partial derivatives of u and v with respect to x and y. Treating y as constant, application of the results already known gives

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x}. \tag{1.18}$$

Exercise 1.23

Write down the corresponding expression for $\frac{\partial h}{\partial y}$.

Example 1.6

Let

$$h(u,v) = u\cos v, \qquad u(x,y) = \sqrt{x^2+y^2}, \qquad v(x,y) = \tan^{-1}\left(\frac{y}{x}\right).$$

Polar coordinates!

Calculate $\partial h/\partial x$ by using the chain rule, assuming $x > 0$.

Solution

First,

$$\frac{\partial h}{\partial u} = \cos v = \frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{\partial h}{\partial v} = -u\sin v = -y,$$

where the assumption $x > 0$ is used in expressing the results in terms of x and y. Also

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}.$$

(The computation of $\partial v/\partial x$ uses the fact that if $z = \tan^{-1} w$ then $dz/dw = 1/(1+w^2)$, a result which is easily checked by differentiating both sides of the equation $\tan z = w$ and using the fact that $\sec^2 z = 1 + \tan^2 z$.) Thus

$$\begin{aligned}\frac{\partial h}{\partial x} &= \left(\frac{x}{\sqrt{x^2+y^2}}\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right) + (-y)\left(-\frac{y}{x^2+y^2}\right)\\ &= \frac{x^2+y^2}{x^2+y^2} = 1.\end{aligned}$$

□

Exercise 1.24

With the same functions as in the example above, calculate $\frac{\partial h}{\partial y}$.

Exercise 1.25

Suppose that h is a function of three variables, u, v and w, each of which is a function of x and y. Write down formulae for the partial derivatives with respect to x and y of the function obtained by substitution for u, v and w in terms of x and y in the expression for h.

Exercise 1.26

Suppose that h is a function of m variables, $u_1, u_2, \ldots, u_m$, each of which is a function of n variables $x_1, x_2, \ldots, x_n$. Write down a formula for the partial derivative with respect to x_i, $i = 1, 2, \ldots, n$, of the function obtained by substituting for the us in terms of the xs in the expression for h.

1.3 Differential equations

As has been pointed out, the study of dynamical systems may be regarded, from one point of view, as the study of systems of differential equations (for continuous systems) or of recurrence relations (for discrete systems). Most of this course deals with continuous systems, and so differential equations will have a very important role. We shall remind you here of some of the fundamental ideas concerned with differential equations that appear in *MST204*.

We shall concentrate on first-order differential equations, that is, equations involving no derivative of the unknown function higher than the first. Our justification for this is that a second- or higher-order equation may always be replaced by an equivalent system of first-order equations, as we shall show later, and on the whole we prefer to deal with first-order systems rather than with higher-order equations in this course.

MST204 Unit 2, Section 1

The most fundamental problem in the theory of first-order differential equations is clearly this: given an equation

$$\frac{dy}{dx} = m(x, y),$$

find all functions $y(x)$ which satisfy the equation, in the sense that when the function $y(x)$ is substituted for the variable y in the expression $m(x, y)$, the result is the derivative dy/dx. (Here $m(x, y)$ is an expression which may involve x and y, but not dy/dx; however, the notation is not supposed to exclude the possibility that either – or indeed both – of these variables is absent.) There are methods for finding solutions analytically, that is to say, for finding formulae for all the solutions in closed form, for differential equations of certain types, and there are a few general principles, but on the whole finding the solutions of differential equations is a very difficult task. If one does not know how to solve a differential equation analytically, one is left with two alternatives: to solve it numerically, or to try to find some qualitative information about the solutions. This course will be concerned entirely with the latter approach.

1.3.1 Analytical methods

Unit 2, Sections 3 and 4

There are two kinds of first-order differential equation for which analytical methods of solution are described in *MST204*. The first consists of those for which the expression $m(x, y)$ can be written as the product of one expression involving x alone and another involving y alone, so that the equation takes the form

$$\frac{dy}{dx} = f(x)g(y).$$

Such equations are said to be separable. The second kind of equation is the linear equation, in which the variable y occurs linearly in the expression $m(x, y)$. In this case the equation takes the form

$$\frac{dy}{dx} = k(x) + l(x)y.$$

Note that in both cases the left-hand side consists of the single term dy/dx; if an equation is not presented initially in this standard form, it must be rearranged before it may be classified and before solution can proceed. The two classes are not mutually exclusive: for example, the equation

$$\frac{dy}{dx} = x^2 y$$

belongs to both.

In practice, a differential equation rarely occurs in a form in which it is immediately obvious whether it falls into either of these classes. Sometimes a transformation of variables can be found which will reduce it to one of these types.

Exercise 1.27

Classify the following equations as separable, linear, both or neither.

(i) $x\dfrac{dy}{dx} = x^2y + 1$ (ii) $y\dfrac{dy}{dx} = x^2y + 1$

(iii) $x\dfrac{dy}{dx} = (x^2 + 1)(y + 1)$ (iv) $y\dfrac{dy}{dx} = x^2 + 1$

Exercise 1.28

How would you recognise an equation which belongs to both classes?

To remind you of the methods of solving these two kinds of equation, we give a couple of examples.

Example 1.7

Find the general solution of the differential equation

$$\frac{dy}{dx} = xe^y,$$

and the particular solution which takes the value 0 when $x = 0$.

Solution

This is a simple example of a separable equation. It may be rewritten as

MST204 Unit 2, Section 3

$$e^{-y}\frac{dy}{dx} = x,$$

and then integration of both sides leads directly to the solution:

$$\int dy\, e^{-y} = \int dx\, x \quad \Longrightarrow \quad -e^{-y} = \tfrac{1}{2}x^2 + c$$
$$\Longrightarrow \quad y = -\ln\left(-\tfrac{1}{2}x^2 - c\right).$$

This is the general solution; c is some constant. To find the particular solution, use the given initial condition to determine the value of c. It is required that $0 = -\ln(0 - c)$, so $c = -1$ and the particular solution is

$$y = -\ln\left(1 - \tfrac{1}{2}x^2\right).$$

The solutions to this equation have two interesting features. Note that in the general solution the constant c is not entirely arbitrary. The equation itself imposes a restriction on it – since the argument of ln must be positive and $-\frac{1}{2}x^2$ is necessarily negative, then c must be negative. Furthermore, each solution is defined only for a limited range of values of x, in fact for $|x| < \sqrt{-2c}$. This feature is one that you will meet again later in the course. □

Example 1.8

Find the general solution of the differential equation

$$\frac{dy}{dx} = y + e^x.$$

Solution

This is a linear equation. The method for solving linear equations given in *MST204* to use an integrating factor. The idea is to rearrange the equation so that all the terms involving y appear on the left-hand side, thus:

Unit 2, Section 4

$$\frac{dy}{dx} - y = e^x;$$

and then to find a function $p(x)$ such that the product of the left-hand side and $p(x)$ is an exact derivative. In this case

$$p(x) = \exp\left(\int dx\,(-1)\right) = e^{-x}.$$

On multiplication throughout by this integrating factor, the equation becomes

$$e^{-x}\left(\frac{dy}{dx} - y\right) = 1.$$

The left-hand side is the derivative of $e^{-x}y$, so

$$\frac{d}{dx}(e^{-x}y) = 1 \quad \Longrightarrow \quad e^{-x}y = x + c \quad \Longrightarrow \quad y = (x+c)e^x,$$

for some constant c. ☐

Exercise 1.29

The equation

$$\frac{dy}{dx} = xy$$

is both separable (for $y \neq 0$) and linear. Solve it by both methods. Is there any difference between the solutions you obtain? What is the particular solution which takes the value $y = 0$ when $x = 0$?

The general solution of a first-order differential equation contains one undetermined constant. Each allowable value of this constant determines a particular solution, that is to say a specific function which satisfies the equation. The usual way of determining a particular solution is by applying an initial condition, as in Example 1.7 and Exercise 1.29.

Linear equations are important, and the concept of linearity of a differential equation applies to more than just first-order equations, as you may recall from *MST204*. An nth-order equation (one involving all orders of derivatives up to and including the nth) is said to be linear if it can be put in the form

Unit 6

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{1.19}$$

where the a_ks and f are given functions of x only, and $a_n(x)$ is not identically zero. The important feature is that y and its derivatives appear linearly, that , there are no products or other complicated expressions involving them. The important consequences of this are twofold. In the first place, the solutions of a homogeneous linear equation – which is one for which the function $f(x)$ vanishes – have the important property of additivity: if $y_1(x)$ and $y_2(x)$ are any two particular solutions of the same linear homogeneous equation, and k_1 and k_2 are any constants, then $k_1y_1(x) + k_2y_2(x)$ is also a solution of the homogeneous equation. In general, there will exist n independent particular solutions of the equation (where n is the degree of the equation), say $y_1(x), y_2(x), \ldots, y_n(x)$, and then

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

the general solution; note that it contains n arbitrary constants, which is the expected number. The property of additivity of solutions is characteristic of linear homogeneous equations, and could be used as an alternative definition of linearity.

The second consequence of linearity applies to the solutions of an inhomogeneous equation – one for which $f(x)$ is not zero. The difference between two solutions of such an equation is always a solution of the corresponding homogeneous equation (the one obtained by replacing $f(x)$ by 0). Thus the general solution of the inhomogeneous equation takes the form $y_p(x) + y_c(x)$, where $y_p(x)$ is a particular solution of the inhomogeneous equation – the particular integral – and $y_c(x)$ is the general solution of the corresponding homogeneous equation – the complementary function.

These results apply even in the first-order case. Consider, for example, the equation

$$\frac{dy}{dx} - y = e^x,$$

which was solved by the integrating factor method in Example 1.8. It is a first-order linear inhomogeneous equation. The general solution is

$$y = (x + c)e^x = xe^x + ce^x.$$

This solution is made up of two parts, as predicted: $y_p = xe^x$ is the particular integral, and $y_c = ce^x$ is the complementary function, that is, the general solution of the corresponding homogeneous equation

$$\frac{dy}{dx} - y = 0.$$

These observations provide a method of solving first-order linear equations which may be preferable to the integrating factor method in some circumstances, particularly when the corresponding homogeneous equation has constant coefficients, as in the case under consideration. Then the solution of the homogeneous equation is easy to find. Moreover, finding a particular integral is often straightforward if one uses the trial and error method described for second-order linear equations in *MST204*. For the case in hand, the inhomogeneous term is e^x, which suggests a first trial solution ke^x; however, this is already the complementary function, so one has to fall back on the expedient of multiplying the obvious trial solution by x, which gives kxe^x as the function to try.

Unit 6, Section 2

Exercise 1.30

Solve each of the equations

$$\frac{dy}{dx} - y = e^x, \qquad \frac{dy}{dx} + y = e^x$$

by the method proposed in the text.

As noted above, despite the fact that this course is about non-linear systems, the properties of linear systems have a very important part to play; this applies in particular to linear differential equations.

1.3.2 Graphical methods

We turn now to the qualitative features of the solutions of differential equations which it may be impossible to solve analytically. In *MST204*, a graphical method is introduced for the representation of a differential equation and as an aid to picturing properties of its solutions. This method forms the basis of our whole approach to the discussion of the qualitative features of differential equations in this course, and is therefore of fundamental importance.

Unit 2, Section 1

The idea is to regard the right-hand side of the first-order differential equation $dy/dx = m(x, y)$ as a direction field. This means that the expression $m(x, y)$ is to be used to calculate a slope at each point of the (x, y)-plane: the values of the x- and y-coordinates are substituted for x and y in $m(x, y)$, and the number so obtained is interpreted as the slope of a line segment through the point, or equivalently as a direction at that point. By carrying out this process repeatedly at different points in the plane, and drawing at each point (x, y) a short line segment in the direction determined by $m(x, y)$, one builds up a graphical picture of the solutions of the differential equation.

Any function $y(x)$ may be represented by its graph in the (x, y)-plane. If $y(x)$ is to be a solution of the differential equation whose direction field has been plotted, then at each point through which its graph passes the slope of the graph must match the slope given by the direction field at the point. It is often possible to gauge by eye how the solutions of the differential equation will run by looking at the direction field, and even to sketch solution curves.

Example 1.9

Draw the direction field of the differential equation solved in Example 1.7, page 20, namely

$$\frac{dy}{dx} = xe^y.$$

Solution

The calculation of the slope at a given point is a simple matter: for this example, with $m(x, y) = xe^y$,

$$m(1,0) = 1 \times e^0 = 1,$$
$$m(1,1) = 1 \times e^1 = e \simeq 2.7.$$

So at the point $(1, 0)$, draw a line segment at 45° to the x-axis, and at $(1, 1)$ a line at $\tan^{-1} 2.7 \simeq 69.5°$. Notice that $m(0, y) = 0$ for every y. The direction field for $-2 \leq x, y \leq 2$ is shown in Figure 1.3; slopes are drawn at the points with integer and half-integer coordinates.

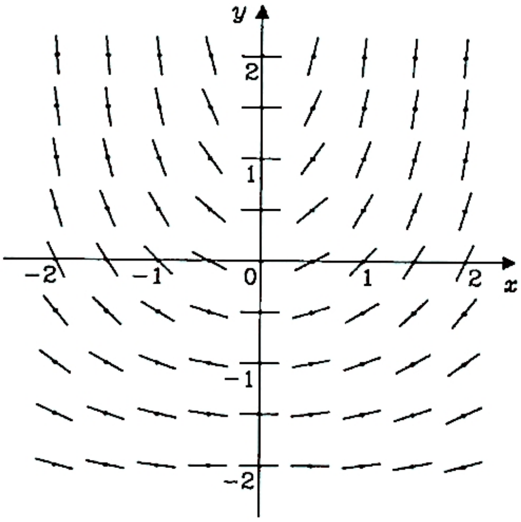

Figure 1.3

t is immediately obvious that the solutions of this differential equation are even functions of x – the graph of a solution must clearly be symmetric with respect to the y-axis. The peculiar feature of solutions, that each is defined on only a finite interval of the x-axis, is also suggested. ☐

Exercise 1.31

Draw the direction field of the differential equation solved in Example 1.8, page 20, namely

$$\frac{dy}{dx} = y + e^x.$$

Use the same ranges and intervals as in Example 1.9.

Techniques based on the idea of sketching the direction field of a differential equation will be used frequently as the course proceeds.

As well as allowing specific differential equations to be pictured, the direction field method throws light on some general results. For example, it is clear immediately that any differential equation has an infinite family of particular solutions. By starting at a point on, say, the y-axis, and following the direction field forward and backward from that point, a particular solution can be generated. This corresponds to specifying a particular solution by its value when $x = 0$. But this discussion makes it clear that specifying a particular solution by its initial value involves an arbitrary choice: there are clearly many ways in which one could specify particular solutions. It is important to note, however, that there is one and only one solution passing through each point in the (x, y)-plane.

1.4 Matrices

1.4.1 Vector notation

In passing, let us make a comment about notation and terminology concerned with vectors. In *MST204* there are three concepts which are closely related, but which use different notation. The first of these is the concept of a vector itself. We use bold type for vectors, thus: **a**, **u**, and so on. A vector in two dimensions, say **a**, may be expressed in terms of two mutually orthogonal unit vectors **i** and **j** in the form

MST204 Unit 14

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}, \qquad (1.20)$$

where a_1 and a_2 are numbers, the components of **a**. This brings us to the second concept, which is that of the components of a vector, or more properly Cartesian components, to make it clear that these are components relative to Cartesian unit vectors. In many texts the components of a vector are written as number pairs: in this notation a vector is written as

$$\mathbf{a} = (a_1, a_2). \qquad (1.21)$$

This is familiar enough in the case of the position vector of a point, $x\,\mathbf{i} + y\,\mathbf{j}$, in which case the corresponding number pair is just the pair of Cartesian coordinates of the point, (x, y). The third possibility is the one used in matrix algebra, in which the vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ is replaced for the purposes of calculation by a column vector. In this context a vector is written as

MST204 Unit 20

$$\mathbf{a} = [a_1 \quad a_2]^T, \qquad (1.22)$$

where the superscript T stands for 'transpose', the operation which converts rows into columns, and is used because it is usually more convenient to write in rows rather than in columns. This course, unlike *MST204*, treats all these concepts on an equal basis, and consequently a symbol such as **a** may stand for a vector expressed in terms of **i** and **j**, a number pair, or a column vector, depending on the context. This may sound as though it could be confusing, but in practice it should not cause any difficulty.

1.4.2 Matrix algebra

There will be little need to use matrices larger than 2 × 2 in this course; but we trust that you can manipulate 2 × 2, 1 × 2 (row) and 2 × 1 (column) matrices confidently and accurately.

All the basic operations of matrix algebra will be employed: matrix addition (especially for column and row matrices), multiplication of a matrix by a scalar (i.e. multiplication of all its entries by the same number), matrix multiplication

MST204 Unit 20, Sections 1 and 2

(i.e. multiplication together of two matrices by the row into column process), and the calculation of inverses of non-singular 2×2 matrices. Use will also be made of the determinants of 2×2 matrices.

MST204 Unit 20,
Section 5

Exercise 1.32

To ensure that you remember how to carry out matrix multiplication, show that if either

$$\mathbf{A}(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

or

$$\mathbf{A}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

then

$$\mathbf{A}(s)\mathbf{A}(t) = \mathbf{A}(t)\mathbf{A}(s) = \mathbf{A}(s+t).$$

Recall that the operations of matrix algebra enjoy most of the properties of the operations of ordinary algebra. For example, matrix multiplication is associative, and is distributive with respect to matrix addition: given any three matrices **A**, **B** and **C** (which are of suitable sizes so that the appropriate sums and products may be formed),

MST204 Unit 20,
Section 2

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C},$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC} \qquad \text{and} \qquad (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}.$$

However, matrix multiplication is *not* commutative: it is not necessarily the case that **BA** and **AB** are equal. It is important always to bear this fact in mind when dealing with matrices.

It is sometimes helpful to concentrate on the columns of a matrix. For 2×2 matrices the notation $\mathbf{A} = [\mathbf{u} \quad \mathbf{v}]$ is convenient; here **u** and **v** are column vectors, each with two entries, **u** being the first column of **A** and **v** the second.

Matrix multiplication on the left is simple to represent in terms of the columns of **A**: if $\mathbf{A} = [\mathbf{u} \quad \mathbf{v}]$ and **B** is another 2×2 matrix, then

$$\mathbf{BA} = [\mathbf{Bu} \quad \mathbf{Bv}]. \tag{1.23}$$

Exercise 1.33

Confirm Equation 1.23 by direct multiplication.

Multiplication of **A** into a column vector corresponds to taking a linear combination of the columns of **A** with the entries in the column vector as coefficients:

$$[\mathbf{u} \quad \mathbf{v}] \begin{bmatrix} a \\ b \end{bmatrix} = a\,\mathbf{u} + b\,\mathbf{v}. \tag{1.24}$$

Exercise 1.34

Confirm Equation 1.24 by direct multiplication.

Exercise 1.35

Show that if

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A} = [\mathbf{u} \quad \mathbf{v}],$$

then

$$\mathbf{u} = \mathbf{Ai} \qquad \text{and} \qquad \mathbf{v} = \mathbf{Aj},$$

and use the distributivity of matrix multiplication to obtain Equation 1.24 in a different way.

Exercise 1.36

Show that

$$[\mathbf{u} \quad \mathbf{v}] \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [a\,\mathbf{u} + c\,\mathbf{v} \quad b\,\mathbf{u} + d\,\mathbf{v}].$$

Two column vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be linearly dependent if there are numbers a and b, not both zero, such that

$$a\,\mathbf{u} + b\,\mathbf{v} = 0.$$

Thus $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if one is a scalar multiple of the other, or if either (or both) of them is zero. If there are no non-zero scalars a and b such that $a\,\mathbf{u} + b\,\mathbf{v} = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are said to be linearly independent. If it is already known that vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then the equation $a\,\mathbf{u} + b\,\mathbf{v} = 0$, regarded as an equation for a and b, can have only one solution: $a = b = 0$. It follows from Exercise 1.34 that if, for some 2×2 matrix $\mathbf{A}$, there is a non-zero column vector $\mathbf{x}$ such that $\mathbf{Ax} = 0$, then the columns of $\mathbf{A}$ must be linearly dependent; and conversely, if the columns of $\mathbf{A}$ are linearly independent, then there can be no non-zero solution to this equation.

1.4.3 Determinants

MST204 Unit 20, Section 5

The determinant, $\det \mathbf{A}$, of the 2×2 matrix $\mathbf{A}$ is the number

$$\det \mathbf{A} = ad - bc, \qquad \text{where} \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The determinant is also written

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

MST204 Unit 20, Section 4

One important use of the determinant is to test whether a matrix has an inverse. To define the inverse of a matrix one first has to know what the unit matrix is. The 2×2 unit matrix, $\mathbf{I}$, is the matrix which has elements 1 on its main diagonal and 0 elsewhere. It has the property, like 1 in ordinary arithmetic, of being multiplicatively neutral: $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for every $\mathbf{A}$. Given a square matrix $\mathbf{A}$, a matrix $\mathbf{A}^{-1}$, if there is one, such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, is called an inverse of $\mathbf{A}$. If $\mathbf{A}$ does have an inverse, and this is by no means necessary, then it is unique. The following statements about a 2×2 matrix $\mathbf{A}$ are all equivalent:

- $\mathbf{A}$ has an inverse;
- the rows of $\mathbf{A}$ are linearly independent;
- the columns of $\mathbf{A}$ are linearly independent;
- there is no non-zero column vector $\mathbf{x}$ such that $\mathbf{Ax} = 0$;
- $\det \mathbf{A} \neq 0$.

A matrix which satisfies any one of these conditions, and therefore all of them, is said to be non-singular; otherwise it is singular. The results of the next two exercises show why the determinant is involved in the question of whether or not a matrix is singular.

Exercise 1.37

Show that $ad - bc = 0$ if and only if either $a = c = 0$ or there is a number k such that $b = ka$ and $d = kc$; and conclude that $\det \mathbf{A} = 0$ if and only if the columns of the matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are linearly dependent. Show similarly that $\det \mathbf{A} = 0$ is a necessary and sufficient condition for the rows of $\mathbf{A}$ to be linearly dependent.

Exercise 1.38

Show that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (\det \mathbf{A})\mathbf{I}.$$

Exercise 1.39

Calculate the determinants of the following matrices:

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \qquad \begin{bmatrix} r\cos\alpha & r\sin\alpha \\ -r\sin\alpha & r\cos\alpha \end{bmatrix}, \qquad \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.$$

Which of them can ever be singular?

The determinant has a number of properties, of which the most important is that it is multiplicative:

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = (\det \mathbf{A})(\det \mathbf{B}). \tag{1.25}$$

Exercise 1.40

Show that if $\mathbf{A}$ is non-singular, then $\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1}$. Show further that for any $\mathbf{A}$ and any non-singular $\mathbf{P}$, $\det(\mathbf{P}^{-1}\mathbf{AP}) = \det \mathbf{A}$.

One useful geometrical interpretation of the determinant of a matrix links it with area. To explain this, it is necessary to make a brief excursion into three-dimensional space.

In *MST204* the cross product, or vector product, of two vectors $\mathbf{a}$ and $\mathbf{b}$ in three-dimensional space is defined as follows:

Unit 14, Section 3

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\mathbf{c}, \tag{1.26}$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the lengths of $\mathbf{a}$ and $\mathbf{b}$, θ is the angle between their directions, and $\mathbf{c}$ is a unit vector at right angles to $\mathbf{a}$ and $\mathbf{b}$ in the direction determined by the right-hand screw rule.

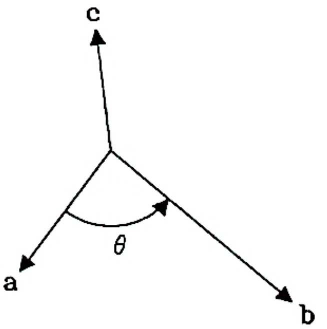

Figure 1.4 The right-hand screw rule.

It was shown that the cross product may be calculated by expressing $\mathbf{a}$ and $\mathbf{b}$ in terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and treating the cross as ordinary multiplication except that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \tag{1.27}$$

(and similar expressions for the other products). If, now, $\mathbf{a}$ and $\mathbf{b}$ both lie in the (x, y)-plane, so that each is a linear combination of $\mathbf{i}$ and $\mathbf{j}$ alone, then the unit vector $\mathbf{c}$ determined by the right-hand screw rule is just $\mathbf{k}$, provided that θ is measured from $\mathbf{a}$ to $\mathbf{b}$ in an anti-clockwise direction. Moreover, if

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} \qquad \text{and} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}, \tag{1.28}$$

then

$$\mathbf{a} \times \mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{k}. \tag{1.29}$$

The coefficient of **k** here is just the determinant of the matrix whose columns are the coefficients of **a** and **b**, namely

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \mathbf{A}.$$

On the other hand, $|\mathbf{a}||\mathbf{b}|\sin\theta$ is the area of the parallelogram shown in Figure 1.5, whose sides are the vectors **a** and **b** – provided that $\theta < \pi$.

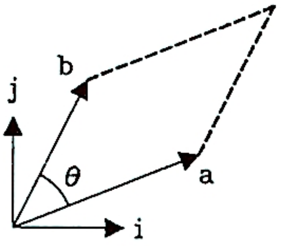

Figure 1.5

Thus det **A** is the area of the parallelogram whose sides are **a** and **b**, provided that the angle measured anti-clockwise from **a** to **b** is less than π. (If not, then det **A** will be the negative of the area. In general, the area of the parallelogram will be the absolute value of the determinant. This slight problem about signs will not be important for this course.)

Exercise 1.41

Let **u** and **v** be a pair of column vectors and **A** a 2×2 matrix. Then **u** and **v** may be regarded as determining vectors in the (x, y)-plane, and **A** as defining a transformation of the (x, y)-plane in which the point whose position vector corresponds to **u** is transformed to the point whose position vector corresponds to **Au**, and similarly for **v**. Show that the area of the parallelogram whose sides are **Au** and **Av** is $|\det \mathbf{A}|$ times the area of the parallelogram whose sides are **u** and **v**.

1.4.4 Eigenvalues and eigenvectors

MST204 Unit 21, Section 1

An eigenvalue of a square matrix **A** is a number λ for which the equation

$$\mathbf{Ax} = \lambda\mathbf{x}$$

has non-zero solutions for the vector **x**. If **A** has an eigenvalue λ, then any non-zero solution **x** of the equation above is an eigenvector corresponding to λ. Notice the use of the word 'any': if **x** is an eigenvector, so is any non-zero scalar multiple of **x**, corresponding to the same eigenvalue. Thus eigenvectors are determined up to scalar multiples at best.

To say that $\mathbf{Ax} = \lambda\mathbf{x}$ is to say that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$; thus in order for λ to be an eigenvalue of **A**, the matrix $\mathbf{A} - \lambda\mathbf{I}$ must be singular. Setting its determinant equal to zero gives a polynomial equation for the eigenvalues of **A**, called the characteristic equation. In the case of a 2×2 matrix, the characteristic equation is a quadratic. Thus **A** may have two distinct real eigenvalues, one real eigenvalue (when the quadratic has coincident roots), or no real eigenvalues at all. The first case is examined in detail in *MST204*; here, the scope is extended to cover the other possibilities.

Unit 21

When the characteristic equation has been solved, the eigenvectors corresponding to any real roots may be found by solving the linear equations $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. The equations will be dependent because of the fact that λ is an eigenvalue, and this ensures that there are non-zero solutions.

The following example will remind you of the process.

Example 1.10

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

Solution

The characteristic equation is

$$\begin{aligned} \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} &= (3-\lambda)(4-\lambda)-2 \\ &= \lambda^2 - 7\lambda + 10 \\ &= (\lambda-2)(\lambda-5) \\ &= 0. \end{aligned}$$

The eigenvalues are thus 2 and 5. The equations for the eigenvectors corresponding to $\lambda = 2$ are

$$\begin{aligned} u + v &= 0, \\ 2u + 2v &= 0. \end{aligned}$$

These are dependent, as they should be, and have the solutions $v = -u$; thus $1 \quad -1]^T$ is an eigenvector, and any other is a non-zero scalar multiple of this one. The eigenvector equations corresponding to $\lambda = 5$ are

$$\begin{aligned} -2u + v &= 0, \\ 2u - v &= 0, \end{aligned}$$

with solutions $[1 \quad 2]^T$ and any non-zero scalar multiple of this. □

Exercise 1.42

Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix}.$$

Exercise 1.43

Show that the characteristic equation of the matrix $\mathbf{A}$ is

$$\lambda^2 - \lambda \operatorname{tr} \mathbf{A} + \det \mathbf{A} = 0,$$

where $\operatorname{tr} \mathbf{A}$, the trace of $\mathbf{A}$, is the sum of the diagonal elements of $\mathbf{A}$.

One use for the eigenvectors of a matrix $\mathbf{A}$, provided there are enough, is to construct another matrix which diagonalises $\mathbf{A}$. If $\mathbf{A}$ has distinct eigenvalues λ_1, λ_2, and if $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to λ_1 and λ_2, then

$$\begin{aligned} \mathbf{A}[\mathbf{x}_1 \quad \mathbf{x}_2] &= [\mathbf{A}\mathbf{x}_1 \quad \mathbf{A}\mathbf{x}_2] \\ &= [\lambda_1\mathbf{x}_1 \quad \lambda_2\mathbf{x}_2] \\ &= [\mathbf{x}_1 \quad \mathbf{x}_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \end{aligned}$$

The column vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, and so the matrix $[\mathbf{x}_1 \quad \mathbf{x}_2]$ is non-singular; denoting it by $\mathbf{P}$ gives

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \qquad (1.30)$$

The diagonal matrix is a standard form for the matrix $\mathbf{A}$.

Exercise 1.44

Consider the real 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(i) Show that the condition that $\mathbf{A}$ has two distinct real eigenvalues λ_1 and λ_2 is

$$\Delta = (\operatorname{tr}\mathbf{A})^2 - 4\det\mathbf{A} = (a-d)^2 + 4bc > 0.$$

(ii) Show that if $b \neq 0$, the two eigenvectors are $[b \quad \lambda_k - a]^T$, $k = 1, 2$.

The argument in this case assumes the existence of two distinct eigenvalues, but in order to develop standard forms for all 2×2 matrices, not just those with distinct eigenvalues, the other possibilities must be dealt with. The case of one eigenvalue is examined next; the case of no real eigenvalues is dealt with in the following subsection.

Suppose that $\mathbf{A}$ has one eigenvalue only, say λ, with a corresponding eigenvector $\mathbf{x}$. Two possibilities arise: there may be another eigenvector $\mathbf{y}$ corresponding to λ, such that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent; or every eigenvector corresponding to λ is a scalar multiple of $\mathbf{x}$.

Exercise 1.45

Consider the matrices

$$\mathbf{A}_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$

Show that they both have just one eigenvalue, namely λ, and that $\mathbf{A}_1$ has two linearly independent eigenvectors while every eigenvector of $\mathbf{A}_2$ is a scalar multiple of $[0 \quad 1]^T$.

In fact, every vector is an eigenvector of $\mathbf{A}_1$; this matrix is a scalar multiple of the identity – it may be written as $\lambda\mathbf{I}$ – and so, for every $\mathbf{u}$,

$$\mathbf{A}_1\mathbf{u} = \lambda\mathbf{u}.$$

Conversely, if a matrix $\mathbf{A}$ has a single eigenvalue λ with two linearly independent eigenvectors $\mathbf{x}$ and $\mathbf{y}$, then it must be a scalar multiple of the identity. To show this, proceed as in the case of two distinct eigenvalues:

$$\begin{aligned} \mathbf{A}[\mathbf{x} \quad \mathbf{y}] &= [\mathbf{A}\mathbf{x} \quad \mathbf{A}\mathbf{y}] \\ &= [\lambda\mathbf{x} \quad \lambda\mathbf{y}] \\ &= [\mathbf{x} \quad \mathbf{y}]\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}. \end{aligned}$$

Thus if $\mathbf{P} = [\mathbf{x} \quad \mathbf{y}]$ then

$$\mathbf{A}\mathbf{P} = \mathbf{P}(\lambda\mathbf{I}).$$

But the identity matrix, and any scalar multiple of it, commutes with any matrix; so

$$\mathbf{A}\mathbf{P} = (\lambda\mathbf{I})\mathbf{P}.$$

The matrix $\mathbf{P}$ is non-singular, since its columns are linearly independent; so multiplying both sides of this equation on the right by $\mathbf{P}^{-1}$ gives

$$\mathbf{A} = \lambda\mathbf{I}.$$

Exercise 1.46

Show that the matrix $\mathbf{A}$ has coincident roots if and only if

$$\Delta = (\operatorname{tr}\mathbf{A})^2 - 4\det\mathbf{A} = (a-d)^2 + 4bc = 0.$$

Show that the common value of the coincident roots is $\frac{1}{2}\operatorname{tr}\mathbf{A}$, and that the components u, v of any eigenvector satisfy the equations

$$\begin{aligned}&(a-d)u + 2bv = 0,\\&2cu - (a-d)v = 0\end{aligned}$$

(which are not independent, because of the condition $(a-d)^2 + 4bc = 0$). Show that there can be two linearly independent eigenvectors only if both of these equations vanish identically, and deduce again that in this case the matrix $\mathbf{A}$ is a multiple of the unit matrix.

It remains to consider the case in which there is only one eigenvector (up to scalar multiples), so that at least one of $(a-d)$, b and c is non-zero. It will be shown that in this case there is a non-singular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ has the form of $\mathbf{A}_2$ in Exercise 1.45, where $\lambda = \frac{1}{2}\operatorname{tr}\mathbf{A}$ is the eigenvalue.

In any eigenvalue problem it may happen that an eigenvalue is zero; it turns out that in the present circumstances the case $\lambda = 0$ is relatively simple to analyse, so that is considered first. In fact, the general case follows easily from this particular one.

Suppose, therefore, that $\mathbf{A}$ has the single eigenvalue 0; then $\operatorname{tr}\mathbf{A} = 0$, and so $\det\mathbf{A} = 0$ also. It follows that the columns of $\mathbf{A}$ are linearly dependent: this may be most conveniently expressed by writing

$$\mathbf{A} = [k_1\mathbf{u} \quad k_2\mathbf{u}],$$

where $\mathbf{u}$ is a non-zero column vector, and k_1 and k_2 are numbers. This formulation allows for the possibility that one or other of the columns of $\mathbf{A}$ is the zero vector (when the corresponding coefficient k_1 or k_2 will be zero); but $\mathbf{A}$ is not the zero matrix (which is the case in which every vector is an eigenvector), and so one at least of k_1 and k_2 must be non-zero. The vector $\mathbf{u}$ is determined only up to a scalar multiple, and this can be taken advantage of to 'normalise' k_1 and k_2: that is, to insist that $k_1^2 + k_2^2 = 1$, or in other words that $\mathbf{k} = [k_1 \quad k_2]^T$ is a unit vector. Now for any column vector $\mathbf{v} = [v_1 \quad v_2]^T$,

$$\mathbf{Av} = v_1(k_1\mathbf{u}) + v_2(k_2\mathbf{u}) = (\mathbf{v}\cdot\mathbf{k})\mathbf{u},$$

which means that the result of operating with $\mathbf{A}$ on any vector $\mathbf{v}$ is a vector which is scalar multiple of $\mathbf{u}$. In particular, $\mathbf{Ak} = \mathbf{u}$ since $\mathbf{k}\cdot\mathbf{k} = 1$. Furthermore, $\mathbf{Av} = \mathbf{0}$ and only if $\mathbf{v}$ is orthogonal to $\mathbf{k}$.

So far, the fact that $\operatorname{tr}\mathbf{A} = 0$ has not been used. The diagonal entries of $\mathbf{A}$ are $_1u_1$ and k_2u_2, so the condition that $\operatorname{tr}\mathbf{A} = 0$ is equivalent to

$$k_1u_1 + k_2u_2 = \mathbf{k}\cdot\mathbf{u} = 0.$$

Thus $\mathbf{u}$ is an eigenvector corresponding to the eigenvalue 0; and so

$$\mathbf{A}[\mathbf{k} \quad \mathbf{u}] = [\mathbf{u} \quad \mathbf{0}] = [\mathbf{k} \quad \mathbf{u}]\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}.$$

The matrix $\mathbf{P} = [\mathbf{k} \quad \mathbf{u}]$ is non-singular, since its columns are non-zero and orthogonal, and so certainly linearly independent; and

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}.$$

This is the required result in the case that the eigenvalue of $\mathbf{A}$ is 0.

the more general case the matrix $\mathbf{A}$ has a single eigenvalue λ, not necessarily zero, but with only one eigenvector (up to scalar multiples). But then the matrix $\mathbf{A} - \lambda\mathbf{I}$ satisfies the conditions of the result already proved: it has the single eigenvalue 0, but does not vanish identically (because $\mathbf{A}$ is not a scalar multiple of the identity). There is therefore a non-singular matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}(\mathbf{A}-\lambda\mathbf{I})\mathbf{P} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

or
$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$

Exercise 1.47

(i) Show that in the general case $\lambda \neq 0$, $\mathbf{k}$ and $\mathbf{u}$, the columns of $\mathbf{P}$, satisfy

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u},$$
$$\mathbf{A}\mathbf{k} = \mathbf{u} + \lambda\,\mathbf{k}.$$

(ii) For the matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad b \neq 0,$$

having a single real eigenvalue, show that we can take

$$\mathbf{u} = [b \quad \tfrac{1}{2}(d-a)]^T \qquad \text{and} \qquad \mathbf{k} = [0 \quad 1]^T.$$

1.4.5 Complex eigenvalues

Example 1.11

Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

Solution

The characteristic equation is

$$(\cos\alpha - \lambda)^2 + \sin^2\alpha = \lambda^2 - 2\lambda\,\cos\alpha + 1 = 0,$$

with complex roots

$$\lambda = \cos\alpha \pm i\sin\alpha = e^{\pm i\alpha}.$$

The vector $[1 \quad i]^T$ is an eigenvector corresponding to $\cos\alpha + i\sin\alpha$; $[1 \quad -i]^T$ is an eigenvector corresponding to $\cos\alpha - i\sin\alpha$. ☐

This would be all very well if complex eigenvalues and eigenvectors were acceptable: but unfortunately real quantities are necessary for our purposes.

Exercise 1.48

Find the (complex) eigenvalues and eigenvectors of

$$\begin{bmatrix} r\cos\alpha & r\sin\alpha \\ -r\sin\alpha & r\cos\alpha \end{bmatrix}.$$

If $\mathbf{A}$ is a real 2×2 matrix (all its entries are real numbers), then its characteristic equation is a real quadratic equation, and so if it has complex roots then the two roots are complex conjugates of each other, which can be written as $\lambda = \mu + i\nu$ and $\bar{\lambda} = \mu - i\nu$. Then there is a complex column vector $\mathbf{x} = \mathbf{u} + i\,\mathbf{v}$ such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \text{or} \qquad \mathbf{A}(\mathbf{u} + i\,\mathbf{v}) = (\mu + i\nu)(\mathbf{u} + i\,\mathbf{v}).$$

Taking the complex conjugate throughout, remembering that $\mathbf{A}$ is real so that $\bar{\mathbf{A}} = \mathbf{A}$, gives

$$\mathbf{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} \qquad \text{or} \qquad \mathbf{A}(\mathbf{u} - i\,\mathbf{v}) = (\mu - i\nu)(\mathbf{u} - i\,\mathbf{v}).$$

Thus if $\mathbf{x}$ is an eigenvector corresponding to λ, then $\bar{\mathbf{x}}$ is an eigenvector corresponding to $\bar{\lambda}$.

Exercise 1.49

Show that $\mathbf{u}$ and $\mathbf{v}$, the real and imaginary parts of the eigenvector $\mathbf{x}$, satisfy

$$\mathbf{A}\mathbf{u} = \mu\,\mathbf{u} - \nu\,\mathbf{v},$$
$$\mathbf{A}\mathbf{v} = \nu\,\mathbf{u} + \mu\,\mathbf{v}.$$

Conclude that

$$\mathbf{A}[\mathbf{u} \quad \mathbf{v}] = [\mathbf{u} \quad \mathbf{v}]\begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}.$$

This is almost enough to prove that when $\mathbf{A}$ has complex eigenvalues $\mu \pm i\nu$ there is a real matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}.$$

The required matrix $\mathbf{P}$ will be $[\mathbf{u} \quad \mathbf{v}]$; however, it still must be shown that this matrix has an inverse, or in other words that it is non-singular. To do so it is sufficient to show that its columns are linearly independent. We shall now see what happens if the columns of the matrix are linearly dependent, so as to rule out this possibility. The columns of the matrix are just the column vectors $\mathbf{u}$ and $\mathbf{v}$; it can be assumed that $\mathbf{v} \neq 0$, otherwise $\mathbf{x} = \mathbf{u}$ would be real, leading to a real eigenvalue λ. If the columns are linearly dependent then $\mathbf{u} = k\mathbf{v}$ for some real number k. But then

$$\begin{aligned} \mathbf{A}(\mathbf{u} + i\,\mathbf{v}) &= (k+i)\mathbf{A}\mathbf{v} \\ &= (\mu + i\nu)(\mathbf{u} + i\,\mathbf{v}) = (k+i)(\mu + i\nu)\mathbf{v}. \end{aligned}$$

The factor $(k+i)$ cancels, leaving

$$\mathbf{A}\mathbf{v} = (\mu + i\nu)\mathbf{v}.$$

But $\mathbf{A}$ and $\mathbf{v}$ are real, so ν must be zero, and $\mathbf{A}$ has a real eigenvalue, with eigenvector $\mathbf{v}$. Since the case of real eigenvalues has already been dealt with, this possibility may be ignored, and it can therefore be assumed that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. The matrix $\mathbf{P} = [\mathbf{u} \quad \mathbf{v}]$ is therefore non-singular, and

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}.$$

Exercise 1.50

Show that two eigenvectors of the real matrix $\mathbf{A}$ having complex eigenvalues are $[b \quad \lambda_k - a]^T$, $k = 1, 2$. Show further that the real and imaginary parts of these vectors are, respectively,

$$[b \quad \tfrac{1}{2}(d-a)]^T \qquad \text{and} \qquad [0 \quad \pm\tfrac{1}{2}\sqrt{|\Delta|}]^T,$$

where $\Delta = (\operatorname{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.

Summary

The results of the last two subsections may be summarised as follows. For every real 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

one can find a real non-singular 2×2 matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ takes one of the following three standard forms:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \qquad \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}. \tag{1.31}$$

Case 1: Real distinct eigenvalues, λ_1, λ_2.

In this case,

$$\Delta = (\operatorname{tr} \mathbf{A})^2 - 4 \det \mathbf{A} = (a-d)^2 + 4bc > 0$$

and **P** is a matrix whose columns are the eigenvectors of **A**,

$$\mathbf{P} = \begin{bmatrix} b & b \\ \lambda_1 - a & \lambda_2 - a \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \qquad (1.32)$$

Case 2: Real equal eigenvalues, $\lambda_1 = \lambda_2 = \lambda$.

In this case,

$$\Delta = (\operatorname{tr} \mathbf{A})^2 - 4 \det \mathbf{A} = (a-d)^2 + 4bc = 0$$

and there are two distinct possibilities.

First, if **A** is diagonal, so that $a = d = \lambda$ and $b = c = 0$, then **A** is already in the standard form, and **P** may be any non-singular matrix.

Second, if **A** is not diagonal, that is not all of $a - d$, b and c are zero, then **A** has only one linearly independent eigenvector **u**, and the second column of **P** is **u** and the first is a vector **k** such that $\mathbf{A}\mathbf{k} = \mathbf{u} + \lambda\mathbf{k}$; for $b \neq 0$ this gives

$$\mathbf{P} = \begin{bmatrix} 0 & b \\ 1 & \frac{1}{2}(d-a) \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}. \qquad (1.33)$$

When $b = 0$, we can take $\mathbf{P} = \text{diagonal}\,[1/c \quad 1]$.

Case 3: Complex eigenvalues, $\lambda = \mu \pm i\nu$.

In this case,

$$\Delta = (\operatorname{tr} \mathbf{A})^2 - 4 \det \mathbf{A} = (a-d)^2 + 4bc < 0$$

and **P** is a matrix whose columns are the real and imaginary parts of the complex eigenvector $\mathbf{z} = \mathbf{u} + i\mathbf{v}$ of **A** satisfying $\mathbf{A}\mathbf{z} = (\mu + i\nu)\mathbf{z}$,

$$\mathbf{P} = \begin{bmatrix} b & 0 \\ \frac{1}{2}(d-a) & \frac{1}{2}\sqrt{|\Delta|} \end{bmatrix} \quad \text{and} \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}. \qquad (1.34)$$

1.4.6 Jacobian matrices

Matrices have all sorts of uses and all sorts of interpretations. One important use of 2 × 2 matrices arises when one considers changes of Cartesian coordinates in the (x, y)-plane. The matrix

MST204 Unit 20,
Section 3

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

represents the effect of a rotation through an angle α of the coordinate axes in an anti-clockwise sense about the origin: the coordinates (x', y') of a point with respect to the rotated axes are obtained from the coordinates (x, y) of the same point with respect to the original axes by the formula

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}; \qquad (1.35)$$

that is, by multiplication by the given matrix. The geometry of this rotation is shown in Figure 1.6.

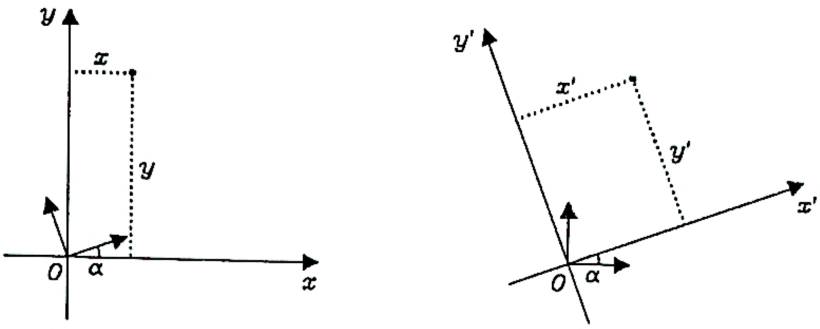

Figure 1.6 Diagram showing the coordinates of a point with respect to different coordinate axes. On the left are shown Cartesian axes Oxy and Cartesian coordinates (x, y), and on the right the Cartesian coordinates (x', y') of the same point with respect to the Cartesian axes $Ox'y'$ obtained by an anti-clockwise rotation about an angle α from Oxy.

Though this possibility is not considered in *MST204*, it is sometimes desirable to use coordinates for the (x, y)-plane based on axes which are not orthogonal and which use different scales, as in Figure 1.7. The transformation of coordinates may then be represented in a similar way to the Cartesian case, but with a matrix which is not a rotation matrix. Such coordinates are called oblique coordinates.

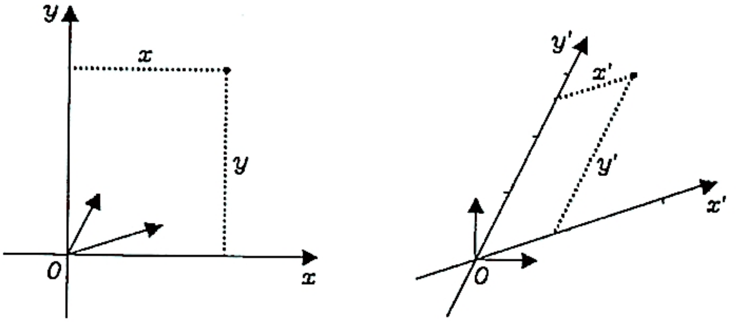

Figure 1.7 On the left are shown the coordinates (x, y) of a point with respect to an orthogonal set of axes Oxy, and on the right the coordinates (x', y') with respect to non-orthogonal axes $Ox'y'$ obtained from Oxy by a linear transformation.

Exercise 1.51

Consider a transformation of coordinates from Cartesian to oblique coordinates given by $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x}$ is the column vector of the Cartesian coordinates of a point, $\mathbf{x}'$ is the column vector of the oblique coordinates of the same point, and $\mathbf{A}$ is a 2×2 matrix. Explain why $\mathbf{A}$ must be non-singular for the oblique coordinates to be well defined. Show that the columns of $\mathbf{A}$ give the oblique coordinates of the points whose Cartesian coordinates are $(1, 0)$ and $(0, 1)$ respectively.

Exercise 1.52

Consider a transformation of coordinates from Cartesian to oblique coordinates given by $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x}$ is the column vector of the Cartesian coordinates of a point, $\mathbf{x}'$ is the column vector of the oblique coordinates of the same point, and $\mathbf{A}$ is the non-singular matrix representing the coordinate transformation. Suppose that $\mathbf{A}$ has a pair of distinct real eigenvalues. What is the significance of the corresponding eigenvectors in terms of the coordinate transformation?

There are many examples of coordinate transformations which cannot be represented by matrices. In the plane, the obvious example is polar coordinates. Other coordinate systems which cannot be obtained from Cartesian ones by matrix transformations are cylindrical and spherical polar coordinates for three-dimensional space. Such coordinate systems are often called curvilinear, for the reason that varying just one coordinate may produce a curve, and not simply a straight line as is the case for oblique (and Cartesian) coordinates. We shall develop some general properties of curvilinear coordinate systems here, for use later in the course.

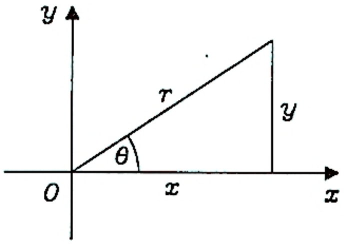

Figure 1.8 Cartesian and polar coordinates.

The relation between the Cartesian coordinates (x, y) of a point and the polar coordinates (r, θ) of the same point is given by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1.36}$$

This may be thought of as defining a map from the (r, θ)-plane to the (x, y)-plane, which specifies the Cartesian coordinates of a point in terms of its polar coordinates, as shown in Figure 1.9.

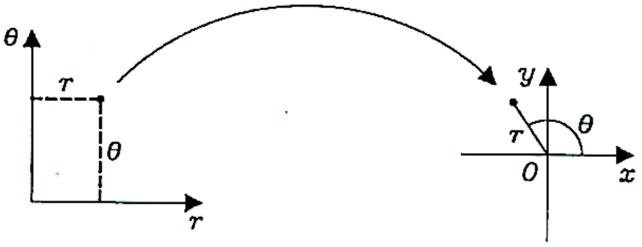

Figure 1.9 The polar coordinate map from the (r, θ)-plane to the (x, y)-plane.

Conversely, the polar coordinates of a point are given in terms of its Cartesian coordinates by

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{1.37}$$

These equations define a map from the (x, y)-plane to the (r, θ)-plane, which is the inverse of the one defined above.

One use of curvilinear coordinates is in surface and volume integrals. Introduction of polar coordinates can sometimes simplify the evaluation of an integral over a region in the plane, and likewise the evaluation of an integral over a region of space may be simplified if the integral is expressed in terms of cylindrical or spherical polars. Areas have an important role to play in this course, and one way of finding an area is to evaluate an integral in which the integrand is the constant function 1.

Thus the area of a region R in the (x, y)-plane may be found by evaluating the double integral

$$\iint_R dx\, dy.$$

The same region R in the (x, y)-plane, when described in terms of polar coordinates, defines a region R' in the (r, θ)-plane. For example, the region of the (x, y)-plane bounded by the arcs of the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and by segments of the x- and y-axes, as shown in Figure 1.10, corresponds to the rectangle $1 \le r \le 2$, $0 \le \theta \le \pi/2$ in the (r, θ)-plane.

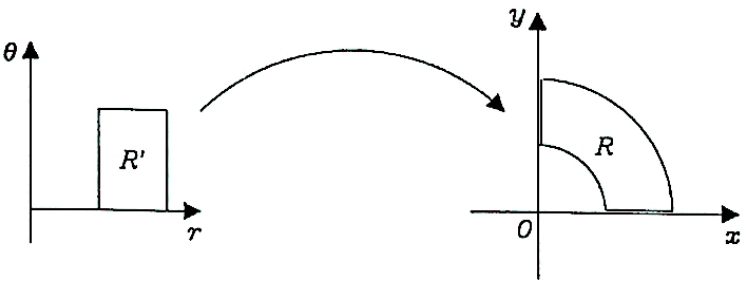

Figure 1.10

In this case the area of R may be computed using polar coordinates, by calculating

$$\int_{\theta=0}^{\pi/2}\int_{r=1}^{2} r\,dr\,d\theta.$$

The factor r has to be included because the area of the region in the (x, y)-plane bounded by the polar coordinate curves with coordinate values r, $r+\delta r$, θ, $\theta+\delta\theta$ is approximately $r\,\delta r\delta\theta$, not $\delta r\delta\theta$. The expression $r\,\delta r\delta\theta$ is often called the element of area in polar coordinates. (We prefer to write $r\,dr\,d\theta$ in the integral, rather than $dr\,d\theta\,r$, because we think of $r\,dr\,d\theta$ as a single indivisible entity.)

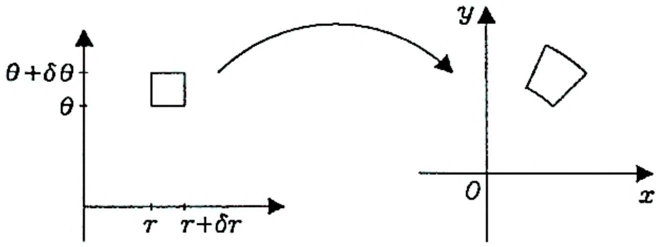

Figure 1.11

The necessity of incorporating the correct element of area in the integral when calculating the area of a region in the (x, y)-plane in terms of polar coordinates may be looked at from a rather different point of view. The (r, θ)-plane may be thought of as a perfectly ordinary two-dimensional space with the usual kind of geometry, with r and θ regarded as Cartesian coordinates in that space, and with area calculated in the usual way. In fact, this view has been taken implicitly in the discussion so far, as is made clear by the figures. The assignment of polar coordinates defines a pair of maps between the (r, θ)-plane and the (x, y)-plane, as explained above. The fact that, for corresponding regions R and R',

$$\iint_R dx\,dy \neq \iint_{R'} dr\,d\theta$$

in general, may be summed up by saying that the polar coordinate maps do not preserve area.

This particular point of view is important, so far as this course is concerned, because we shall be interested in maps between two-dimensional spaces similar to the polar coordinate map; but we shall be interested principally in those which do preserve area. It will be necessary therefore to develop a test for whether or not a map does preserve area; we tackle this task now. The basic idea is to develop a systematic way of computing the element of area in general curvilinear coordinates; this will involve the use of a determinant to measure ratios of areas.

In general, the introduction of curvilinear coordinates, say (u, v), will be specified by a map from the (u, v)-plane to the (x, y)-plane, which gives the Cartesian coordinates of a point in terms of the curvilinear coordinates by means of two functions of two variables

$$x = x(u, v), \qquad y = y(u, v).$$

It must, of course, be possible to solve for the curvilinear coordinates in terms of the Cartesian ones, so that u and v must be expressible in terms of x and y in the form

$$u = u(x,y), \qquad v = v(x,y).$$

This represents a map from the (x,y)-plane to the (u,v)-plane, which is inverse to the map at the start.

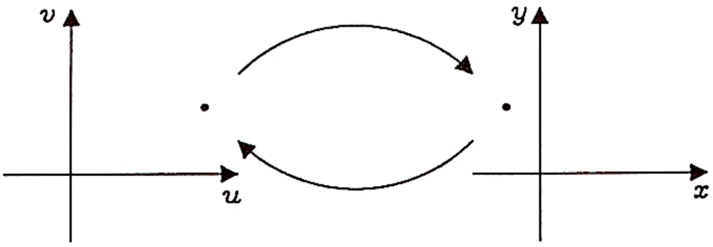

Figure 1.12

Consider the rectangle in the (u,v)-plane whose vertices have coordinates (u,v), $(u+\delta u, v)$, $(u, v+\delta v)$ and $(u+\delta u, v+\delta v)$. Now calculate the images of these points in the (x,y)-plane, working to first order in δu and δv, which are taken to be small.

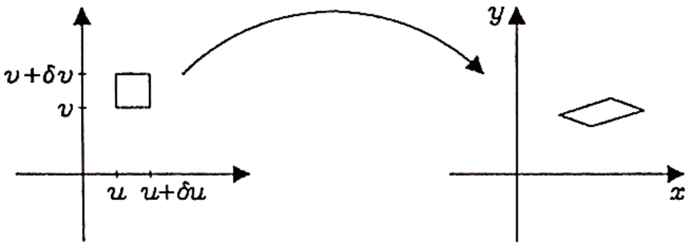

Figure 1.13

The first-order Taylor approximation gives

Equation 1.5, page 14

$$\begin{aligned}
x(u+\delta u, v) &\simeq x(u,v) + \frac{\partial x}{\partial u}\delta u,\\
x(u, v+\delta v) &\simeq x(u,v) + \frac{\partial x}{\partial v}\delta v,\\
x(u+\delta u, v+\delta v) &\simeq x(u,v) + \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v,
\end{aligned}$$

and

$$\begin{aligned}
y(u+\delta u, v) &\simeq y(u,v) + \frac{\partial y}{\partial u}\delta u,\\
y(u, v+\delta v) &\simeq y(u,v) + \frac{\partial y}{\partial v}\delta v,\\
y(u+\delta u, v+\delta v) &\simeq y(u,v) + \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v.
\end{aligned}$$

Thus, to this approximation, the images in the (x,y)-plane of the vertices of the rectangle are the vertices of a parallelogram with one vertex at $(x,y) = (x(u,v), y(u,v))$ and with sides determined by the vectors

$$\begin{bmatrix} \dfrac{\partial x}{\partial u}\delta u \\ \dfrac{\partial y}{\partial u}\delta u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dfrac{\partial x}{\partial v}\delta v \\ \dfrac{\partial y}{\partial v}\delta v \end{bmatrix}.$$

The area of this parallelogram is given, up to a sign, by

$$\begin{aligned}
\begin{vmatrix} \dfrac{\partial x}{\partial u}\delta u & \dfrac{\partial x}{\partial v}\delta v \\ \dfrac{\partial y}{\partial u}\delta u & \dfrac{\partial y}{\partial v}\delta v \end{vmatrix} &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \delta u\,\delta v \\
&= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \delta u\,\delta v.
\end{aligned}$$

Thus the element of area in the Oxy plane is given by

$$\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|\delta u\,\delta v.$$

The matrix of partial derivatives

$$\begin{bmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{bmatrix} \tag{1.38}$$

is called the *Jacobian matrix* of the map from (u, v) to (x, y), and its determinant simply the *Jacobian* of the map. The Jacobian (i.e. the determinant) is denoted by

Jacobian matrix

Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)}=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right).$$

So the element of area in the (x, y)-plane is obtained by multiplying $\delta u\,\delta v$, the element of area in the (u, v)-plane, by $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$. For maps with positive Jacobian – these are the ones which do not change right-handed to left-handed coordinates – the element of area in the (x, y)-plane, in terms of the curvilinear coordinates u, v, is just

$$\delta x\,\delta y=\frac{\partial(x,y)}{\partial(u,v)}\delta u\,\delta v. \tag{1.39}$$

Notice that this is the two-dimensional analogue of the relation

$$\delta x=\frac{dx}{du}\delta u,$$

where $x(u)$ is a function of u alone. In this one-dimensional case, provided $dx/du \neq 0$ we may invert the relationship to give u as a function of x; then

$$\frac{dx}{du}\frac{du}{dx}=1. \tag{1.40}$$

Similarly, in the two-dimensional case we may consider (u, v) as functions of (x, y), and by precisely the same analysis as above obtain

$$\delta u\,\delta v=\frac{\partial(u,v)}{\partial(x,y)}\delta x\,\delta y.$$

Thus, by comparison with Equation 1.39, we obtain the analogous relation to 1.40,

$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}=1.$$

This relation is proved formally below.

Example 1.12

Plane polar coordinates

$$x=r\cos\theta,\qquad y=r\sin\theta,$$

give

$$\frac{\partial x}{\partial r}=\cos\theta,\qquad \frac{\partial x}{\partial\theta}=-r\sin\theta,$$
$$\frac{\partial y}{\partial r}=\sin\theta,\qquad \frac{\partial y}{\partial\theta}=r\cos\theta.$$

Thus

$$\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta}\\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r,$$

confirming that the element of area is $r\,\delta r\,\delta\theta$. ☐

Exercise 1.53

Find the element of area in the (x, y)-plane for the curvilinear coordinates defined by

$$x(u,v) = \sqrt{u}\cos v, \qquad y(u,v) = \sqrt{u}\sin v \qquad (\text{where } u > 0).$$

Exercise 1.54

Show that for a linear change of coordinates, for example to oblique coordinates, given by $\mathbf{x}' = \mathbf{A}\mathbf{x}$, the element of area is $|(\det \mathbf{A})^{-1}|\,\delta x'\delta y'$. Confirm that rotations preserve area.

This approach to the question of the change of area resulting from a change of coordinates allows the treatment of both sets of coordinates essentially on an equal footing, assuming as always that the two systems of coordinates are interchangeable, in the sense that each can be solved for the other. Thus in the case of plane polar coordinates one may deal as well with the coordinate transformation which gives Cartesians in terms of polars:

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Exercise 1.55

Find $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

Show that, provided that (x, y) and (r, θ) are the Cartesian and polar coordinates of the same point,

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)^{-1}.$$

The second result of this exercise is an instance of a general rule, already suggested above, whose formal derivation follows. Suppose that the equations

$$x = x(u,v), \qquad y = y(u,v)$$

may be solved to give u and v in terms of x and y, or in other words that an invertible coordinate transformation is involved, as explained above. This means that there are functions $u(x, y)$ and $v(x, y)$ such that

$$x(u(x,y), v(x,y)) = x, \qquad y(u(x,y), v(x,y)) = y;$$

these functions specify the inverse coordinate transformation. The equations above may be partially differentiated with respect to x and y. The chain rule may be applied to their left-hand sides. For the right-hand sides, note simply that

$$\frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = 1, \qquad \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0.$$

Then

$$\begin{aligned}
&\frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} = 1,\\
&\frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} = 0,\\
&\frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = 0,\\
&\frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} = 1.
\end{aligned}$$

These equations may be written more succinctly as a single matrix equation:

$$\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus the Jacobian matrices of the two inverse coordinate transformations are inverses of each other. The determinant of the product of two matrices is the product of the determinants of the factors, so

$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} = 1.$$

(In all these calculations it must be remembered that everything is evaluated at points in the (u,v)-plane and the (x,y)-plane whose coordinates correspond under the coordinate transformations.)

Note that the Jacobian matrix of a map from the (u,v)-plane to the (x,y)-plane, or vice versa, which is invertible, is necessarily a non-singular matrix. This result has the following converse: if the map from, say, the (u,v)-plane to the (x,y)-plane has a non-singular Jacobian matrix, then close enough to any image point in the (x,y)-plane the map may be inverted; that is to say, the formulae for x and y in terms of u and v may be solved to give u and v in terms of x and y. The test for non-singularity is of course that the Jacobian determinant should be non-zero. The corresponding result for a function of a single variable is this: if f, a function of one variable, has non-vanishing derivative then it may be inverted, in the sense that the equation $f(x) = y$ may be solved to give x in terms of y. Thus the non-vanishing of the Jacobian for maps between two-dimensional spaces plays a similar role, as a test for the existence of an inverse, to the non-vanishing of the derivative for a function of one variable (a map between one-dimensional spaces).

Exercise 1.56

As another application of the Jacobian matrix, show that if a curve is given parametrically in the form $\mathbf{x}(t)$ in Cartesian coordinates, and $\mathbf{u}(t)$ in curvilinear coordinates, then the components of its tangent vector in terms of the two coordinate systems are related by

$$\frac{d\mathbf{x}}{dt} = \mathbf{J}\frac{d\mathbf{u}}{dt},$$

where $\mathbf{J}$ is the Jacobian matrix. Show also that, if $\mathbf{e}_u$ is the tangent vector to the curve $(x(u+t,v), y(u+t,v))$ in the (x,y)-plane – a u-coordinate curve – and $\mathbf{e}_v$ is the tangent vector to the curve $(x(u,v+t), y(u,v+t))$ – a v-coordinate curve – then

$$\frac{d\mathbf{x}}{dt} = \frac{du}{dt}\mathbf{e}_u + \frac{dv}{dt}\mathbf{e}_v.$$

This has taken us a little way away from our original concern, which was with maps (or coordinate transformations) which preserve area. This section ends by stating the condition for a map from the (u,v)-plane to the (x,y)-plane to preserve area: the map must have constant Jacobian equal to 1, that is,

$$\frac{\partial(x,y)}{\partial(u,v)} = 1.$$

Exercise 1.57

By modifying a map given in Exercise 1.53, find an example of a map from the (u,v)-plane to the (x,y)-plane which preserves area.

1.5 Systems of differential equations

By a system of differential equations is meant a set of first-order differential equations for several unknown functions, say n equations for the functions $x_1(t), x_2(t), \ldots, x_n(t)$, of the form

$$\begin{aligned}
\dot{x}_1 &= \frac{dx_1}{dt} = v_1(x_1, x_2, \ldots, x_n, t), \\
\dot{x}_2 &= \frac{dx_2}{dt} = v_2(x_1, x_2, \ldots, x_n, t), \\
&\vdots \\
\dot{x}_n &= \frac{dx_n}{dt} = v_n(x_1, x_2, \ldots, x_n, t),
\end{aligned}$$

where the functions $v_1, v_2, \ldots, v_n$ may be functions of all the dependent variables $x_1, x_2, \ldots, x_n$ and the independent variable t. (The overdot will also be used to denote differentiation with respect to t.) In *MST204*, the case of a linear first-order constant-coefficient system is extensively discussed. This is a system for which the functions $v_1, v_2, \ldots, v_n$ are linear functions of the xs with coefficients which are constant, except possibly for an inhomogeneous term. It is natural to adopt a vectorial notation when dealing with systems of differential equations; the general case may be written

Unit 22

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}, t),$$

while the special case dealt with in *MST204* takes the form

Unit 22, Section 1

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{k}(t),$$

where $\mathbf{A}$ is a constant matrix, but the column vector $\mathbf{k}(t)$ which contains the inhomogeneous terms need not be constant.

A solution of a system of first-order differential equations in n unknowns is a set of n functions $x_1(t), x_2(t), \ldots, x_n(t)$ with the property that, when they are substituted for the variables $x_1, x_2, \ldots, x_n$ on the right-hand sides of the equations, the results are the derivatives on the left-hand sides. As mentioned in Section 1.3, solving single differential equations is difficult enough. The problem of solving a system of equations is much more complicated, in general, because the equations are usually too closely interrelated with each other to allow one to separate out the variables.

There is a systematic approach to the solution of linear constant-coefficient systems of equations, which is discussed at length for some cases of systems in two variables in *MST204*, and whose main features are recounted below. In other cases we are reduced to searching for qualitative information about the solutions; a modification of the method of direction fields will be our main tool for this purpose.

Unit 22

There are a couple of general points worth making about the solutions of a system of first-order equations. Since solution of differential equations involves integration, in principle if not always in practice, undetermined constants must be expected in the solutions. For a system of first-order equations in n unknowns it would be natural to expect there to be one constant for each unknown. This will certainly be so if the equations are separate, in the sense that each involves only one of the dependent variables, so that solving the system reduces to solving the individual equations one by one. In fact, it is true in general: if there are n unknowns then there will be n undetermined constants in the general solution. To determine a particular solution completely, the value of each of these constants must be specified. This will usually be done by applying initial conditions: the value of each of the variables $x_1, x_2, \ldots, x_n$ at $t = 0$ (or some other convenient value of t) will be specified in advance, and this information will then be used to calculate the values of the undetermined constants in the general solution. The basic principle here is that a particular solution of a given system of first-order equations is completely determined (for all values of t, positive and negative) by the values of the xs when $t = 0$. This seemingly innocuous point is actually fundamental to the use of systems of differential equations to represent dynamical systems.

Consider next the solution of the linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{k}(t).$$

As in the case of a single linear equation, linearity has two important consequences. First, for a homogeneous system, which is one for which $\mathbf{k}(t) \equiv \mathbf{0}$, the solutions have the additivity property: if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are particular solutions and c_1 and c_2 are constants, then $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is also a solution. (Here one has to remember that the solutions are being represented as column vectors of functions, and the algebraic operations implied are those of column vector or matrix algebra.)

Exercise 1.58

Confirm that this is the case.

Secondly, the general solution of the inhomogeneous system takes the form $\mathbf{x}_p(t) + \mathbf{x}_c(t)$, where $\mathbf{x}_p(t)$ is any one particular solution – the particular integral – and $\mathbf{x}_c(t)$ is the general solution of the homogeneous system – the complementary function.

To try to find the general solution of the homogeneous system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

one may proceed in one of two ways. As a first attempt, guided by the fact that solutions of homogeneous linear differential equations with constant coefficients often seem to involve exponentials, one makes a trial solution of the form

$$\mathbf{x}(t) = \mathbf{a}e^{\lambda t}.$$

Here $\mathbf{a}$ is a constant column vector, and

$$\text{if}\quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad \text{then}\quad \mathbf{a}e^{\lambda t} = \begin{bmatrix} a_1e^{\lambda t} \\ a_2e^{\lambda t} \\ \vdots \\ a_ne^{\lambda t} \end{bmatrix}.$$

Thus $\dot{\mathbf{x}}(t) = \lambda\mathbf{a}e^{\lambda t}$, and $\mathbf{x}(t)$ is a solution if

$$\lambda\mathbf{a}e^{\lambda t} = \mathbf{A}\mathbf{a}e^{\lambda t}.$$

This can be the case only if

$$\mathbf{A}\mathbf{a} = \lambda\mathbf{a}.$$

Thus if $\mathbf{a}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{a}e^{\lambda t}$ is a solution of the system; and this is the only possibility, for non-zero $\mathbf{a}$. If the matrix $\mathbf{A}$ has n linearly independent real eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$C_1\mathbf{a}_1e^{\lambda_1 t} + C_2\mathbf{a}_2e^{\lambda_2 t} + \cdots + C_n\mathbf{a}_ne^{\lambda_n t}$$

is a solution for arbitrary constants $C_1, C_2, \ldots, C_n$, and since it contains n arbitrary constants, it is the general solution.

So far so good; but of course not every matrix has the requisite number of linearly independent real eigenvectors. So an alternative approach may be required. We shall describe this for the case $n = 2$ only, which is sufficient for the purposes of this course; moreover, the method is based on the method for the solution of inhomogeneous equations which is discussed in *MST204*, but only for systems with two unknowns.

Unit 22, Section 2

We continue to study the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

but assume now that there are just two equations, for two dependent variables, so that $\mathbf{A}$ is a constant 2×2 matrix. The idea is to use the results of Subsections 1.4.3 and 1.4.4 to convert the matrix $\mathbf{A}$ to the appropriate standard form, by making in effect a transformation of coordinates. Suppose that $\mathbf{P}$ is a constant matrix such that $\mathbf{P}^{-1}\mathbf{AP}$ is one of the standard forms given in the summary at the end of Subsection 1.4.4. (Which standard form occurs will depend, of course, on what eigenvalues and eigenvectors $\mathbf{A}$ has.) Define new variables $\mathbf{y} = [y_1 \quad y_2]^T$ by $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, or implicitly by $\mathbf{x} = \mathbf{Py}$. Then $\mathbf{y}$ satisfies

$$\dot{\mathbf{y}} = \mathbf{P}^{-1}\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{APy}.$$

The original system is solved by solving the new one for $\mathbf{y}$ and converting back using the formula $\mathbf{x} = \mathbf{Py}$. This means that if the system of equations can be solved when the coefficient matrix is in standard form, then any system of this type can be solved. Moreover, and this is perhaps the real advantage of this approach, it can be seen that the qualitative features of the solution for a general coefficient matrix will be essentially the same as those of the solution for the corresponding standard form, since one is obtained from the other simply by a linear transformation of coordinates. It remains therefore to find the solutions of the system for each of the standard forms of 2 x 2 matrices. The analysis and description of the solutions is carried out in detail in *Unit 3*; here we deal just with the derivation of the solutions, mostly in the form of exercises.

Exercise 1.59

Show that if the coefficient matrix $\mathbf{B}$ of the system $\dot{\mathbf{y}} = \mathbf{By}$ is diagonal, with entries λ_1 and λ_2 along the diagonal, then the system consists of two completely separate equations,

$$\dot{y}_1 = \lambda_1 y_1, \qquad \dot{y}_2 = \lambda_2 y_2.$$

Solve these equations, and confirm that the results of this approach agree with those of the exponential trial solution method discussed earlier.

Exercise 1.60

Suppose next that

$$\mathbf{B} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$

Show that in this case the equations separate partially, in the sense that the equation for y_1 does not contain y_2. Use this fact to solve the equations. (You should find the results of Example 1.8 of Subsection 1.3.1 useful.)

The final case, when the matrix has complex eigenvalues, is less easy to solve. However, it is a reasonable guess that the solutions will involve sines and cosines in this case, as often happens when one is dealing with complex roots, and sometimes they may be found by trial using this observation, as in the next exercise. A more elegant complete method of solution is described in *Unit 3*, and another complete but less elegant method may be found in Exercise 1.62 below.

Exercise 1.61

Suppose that

$$\mathbf{B} = \begin{bmatrix} 0 & \nu \\ -\nu & 0 \end{bmatrix},$$

which is the standard form for matrices with purely imaginary eigenvalues. Find the general solution of the system $\dot{\mathbf{y}} = \mathbf{By}$.

There is an interesting relationship between constant-coefficient homogeneous linear first-order systems in two dependent variables and constant-coefficient homogeneous linear second-order systems in a single dependent variable. We include an investigation of this relationship in the form of an exercise. To some extent this is a retrograde step, in view of our expressed intention of replacing second-order equations by systems of first-order ones; however, it does provide a further insight into how the various methods of solution of constant-coefficient linear differential equations are related.

Exercise 1.62

Consider the general case of a constant-coefficient homogeneous linear first-order system in two variables, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The equations are therefore

$$\dot{x}_1 = ax_1 + bx_2,$$
$$\dot{x}_2 = cx_1 + dx_2.$$

(i) By differentiating the first of these again, and using the original pair of equations to substitute for x_2 and $\dot{x}_2$, show that x_1 satisfies a constant-coefficient homogeneous linear second-order differential equation. Show that x_2 satisfies the same equation.

(ii) Show that the auxiliary equation of the second-order equation you have just found is the characteristic equation of the matrix $\mathbf{A}$.

(iii) Use this approach to confirm again the general solution of the system when $\mathbf{A}$ has distinct real eigenvalues.

(iv) Use this approach to find the general solution of the system when

$$\mathbf{A} = \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}.$$

1.6 Newtonian mechanics

1.6.1 Motion of a particle along a straight line

We begin by considering the simplest kind of dynamical system, which consists of a single particle moving along a straight line under the action of a force or several forces.

MST204 Unit 4

The first step in analysing the motion of a particle along a straight line – or indeed any kind of motion – is to decide how the state of the particle is to be defined. As in *MST204*, the line concerned is turned into an x-axis, by choosing an origin, a direction to count as positive, and a unit of length, so that each point on the line may be labelled by a real number x, such that $|x|$ is the distance of the point from the origin. It is worth noting in passing that this process assigns to each point on the line a unique coordinate, to distinct points distinct coordinates, and to nearby points nearby coordinates. These are the essential requisites of a coordinate system for the whole line; in one sense of the term, the fact that the line can be described as a 'one-dimensional' space simply recognises that each of its points may be labelled by one coordinate in this unambiguous way. More generally, if each point of some space can be labelled by a single coordinate in a similar fashion then the space is said to be one-dimensional.

Unit 4, Section 1

Now return to the particle moving along the straight line. Once time has been given coordinates, in a similar fashion to the spatial line, the position of the particle at any instant of time will specify its position function $x(t)$. The particle's (instantaneous) velocity $v(t)$ at time t is the derivative of the position function,

$$v(t) = \frac{dx}{dt}(t),$$

which is often written $\dot{x}(t)$. The state of the system at time t is determined by both its position $x(t)$ and its velocity $v(t) = \dot{x}(t)$. Thus the phase space is a two-dimensional space with coordinates x and v.

The basic problem of Newtonian mechanics is to determine the particle's position in terms of time, that is, to find $x(t)$ explicitly, from a knowledge of the forces acting on the particle and of its initial state. The essential ingredient in the solution of this problem is Newton's second law, which relates the total force acting on the particle to the kinematical variables x and v. This law states that

$$ma = F,$$

where m is the particle's mass, assumed constant, a is its acceleration and F is the total force acting on it. The acceleration is just the rate of change of velocity: $a(t) = \dot{v}(t) = \ddot{x}(t)$. Newton's second law thus leads to a second-order differential equation for x, and once this equation has been set up, the solution of the dynamical problem is reduced to solving the equation. Newton's second law involves the particle's acceleration, and therefore leads to a second-order equation, making it necessary to specify both position and velocity to determine the state of a mechanical system.

When setting up the equation of motion using Newton's second law, one has to exercise care in computing the total force. The total force is the vector sum of all the individual forces acting, and in taking the sum, due regard must be paid to the directions as well as the magnitudes of the forces acting.

The following examples and exercises should remind you of the process of setting up and solving equations of motion for rectilinear motion.

Example 1.13

Find the motion of a particle subject to a constant force (for example, motion under terrestrial gravity).

Solution

Let the value of the force be F_0, a constant, positive or negative according as the force acts in the positive or negative x direction. The equation of motion is

$$m\ddot{x} = F_0, \qquad (1.41)$$

and this is easily solved (by direct integration) to give

$$x(t) = \frac{F_0}{2m}t^2 + v_0 t + x_0, \qquad (1.42)$$

where v_0 is the particle's initial velocity and x_0 is its initial position, so that (x_0, v_0) is its initial state (its state at $t = 0$). □

Exercise 1.63

Describe the motion of the particle of Example 1.13 for $t > 0$ if

(i) v_0 and F_0 have the same sign,

(ii) v_0 and F_0 have opposite signs.

Example 1.14

Suppose that, in addition to a constant force as in the previous example, the particle is acted on by a force whose value is proportional to the particle's position. Write down the equation of motion.

Solution

Let the value of the constant force be F_0 as before. The value of the additional force will be kx for some constant k. The value of the total force is thus $F_0 + kx$ and the equation of motion is

$$m\ddot{x} = F_0 + kx.$$

□

Exercise 1.64

Show that, in the situation specified in Example 1.14, the force proportional to position will either always be directed towards the origin (the attractive case) or always be directed away from the origin (the repulsive case), depending on the sign of k. Show that, provided that $k \neq 0$, there will always be a position of equilibrium, that is, a point at which the particle can remain at rest indefinitely. Show that in the attractive case the motion is given by

$$x(t) = -\frac{F_0}{k} + A\cos\omega t + B\sin\omega t, \qquad \text{where} \quad \omega^2 = -\frac{k}{m},$$

and identify the constants A and B in terms of the initial state of the particle. Find the motion in the repulsive case.

1.6.2 Energy

MST204 Unit 4, Section 4

In straight-line motion, if the (total) force is a function of position only, so that it may be written $F(x)$, then the quantity

$$E = \tfrac{1}{2}mv^2 - \int dx\, F(x) = T(v) + U(x) \tag{1.43}$$

remains constant throughout the motion. This quantity E is called the energy (strictly the total mechanical energy) of the particle; of its component parts, $T = \frac{1}{2}mv^2$ is called the kinetic energy and U the potential energy. The potential energy is defined in terms of the indefinite integral of the force:

$$U(x) = -\int dx\, F(x). \tag{1.44}$$

Alternatively, if the potential energy U is given, the force may be found by differentiation:

$$F(x) = -\frac{dU}{dx}. \tag{1.45}$$

The potential energy is not uniquely defined, since the indefinite integral is determined only up to the addition of a constant; or again, two functions U and V which differ only by a constant have the same derivative and therefore define the same force if they are taken as potential energies. However, this indeterminacy turns out to have no significance: in general, it is only differences in potential energy that matter, and addition of a constant to the potential energy is therefore immaterial. There is no absolute significance to the zero level of potential energy (or of total energy, for that matter); in practice one often takes advantage of this to choose a zero level which is convenient for the problem in hand.

Strictly speaking, one should talk of *a* potential energy *function* for a given force, rather than of the potential energy of the force, and to do so would resolve some of the confusions which may possibly arise from the indeterminacy which has just been discussed; however, the less precise usage is so standard that we shall adopt it.

Exercise 1.65

Find the potential energy of the force in Example 1.14, taking $x = 0$ as zero level.

In straight-line motion, the concepts of position-dependent force and of potential energy are interchangeable: the law of motion may be formulated equally well in terms of either. In *MST204* mechanical problems are usually discussed in terms of forces; in this course, on the other hand, it turns out to be more satisfactory to give pride of place to potential energy. A dynamical system is usually specified by giving the potential energy, usually denoted by V, and the equation of motion is taken to be

$$m\dot{v} = -\frac{dV}{dx}. \tag{1.46}$$

1.6.3 Momentum

The linear momentum of a particle of mass m and velocity v is defined to be the product mv. In *MST204* the momentum is used mainly in problems involving collisions, where the application of the law of conservation of linear momentum enables one to find information about the speeds and directions of motion of billiard balls (for example) after a collision, from the corresponding information for the balls before the collision. In that context, since the problems involve motion in two (or three) dimensions, the momentum is a vector quantity; for one-dimensional motion the momentum, though still strictly speaking vectorial, may be regarded as a scalar.

Unit 17, Section 2

This course has nothing to say about collisions. Even so, momentum will play an important role; but simply as a convenient way of expressing the equation of motion. If the symbol p is used for the momentum, then its definition may be written in the form

$$\dot{x} = \frac{p}{m} \qquad (1.47)$$

and the equation of motion as

$$\dot{p} = -\frac{dV}{dx}. \qquad (1.48)$$

The equation of motion derived from Newton's second law, which in *MST204* was always treated as a single second-order differential equation, has thus been converted into a system of two first-order differential equations. Moreover, since p is simply related to v, one can as well use x and p to specify the state of the system as x and v. In fact, this system of first-order equations provides a convenient phase space representation of the system. It will steadily become more and more apparent as the course progresses that there are considerable conceptual benefits to be gained from this simple change of representation.

Exercise 1.66

Express the total mechanical energy in terms of the momentum rather than the velocity.

1.6.4 Circular motion

Although this course is concerned only with motion of particles in one dimension, attention is not restricted to motion along a straight line. A particle constrained to move on a curve, such as a circle, is an example of a system in a one-dimensional space, in the sense that the particle's position is fixed if the value of just one coordinate is known – say the angle which the radius from the centre of the circle to the particle makes with a fixed radius.

There are several ways in which the particle may be constrained to stay on the circle: it may be attached to the centre by a light inextensible string; it may be a bead strung onto a smooth circular wire; it may be held in place by gravity, like a toy car on a frictionless 'hot-wheels' track (provided that it is going fast enough for its height). In each case there is a force associated with the constraint: the tension in the string, or the reaction of the wire on the bead, or of the track on the toy car. This force does not act along the direction of the motion, but it does affect the motion of the particle; it may appear therefore that the system is essentially more complicated than a particle moving along a straight line. However, in each case the direction of the force of constraint is perpendicular to the circle (pointing towards the centre, in other words) – in the case of the bead and toy car because of the assumption that there is no friction. Under these circumstances, the equations of motion can be formulated without reference to the force of constraint.

Suppose that the particle has mass m, the circle radius R, the force of constraint magnitude T, directed towards the centre of the circle, and that there is a total force **F** acting on the particle, in addition to the constraint force, in the plane of the circle. The position of the particle is determined by the angle θ. Choose axes as shown.

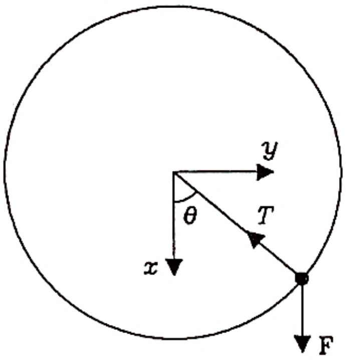

Figure 1.14

The components of the particle's acceleration may be derived as follows. The particle's position vector **r** relative to the centre of the circle is given by

$$\mathbf{r} = R\cos\theta\,\mathbf{i} + R\sin\theta\,\mathbf{j}. \tag{1.49}$$

Thus

$$\dot{\mathbf{r}} = -(R\sin\theta)\dot{\theta}\,\mathbf{i} + (R\cos\theta)\dot{\theta}\,\mathbf{j}$$

and

$$\begin{aligned}\ddot{\mathbf{r}} &= -\left((R\sin\theta)\ddot{\theta} + (R\cos\theta)\dot{\theta}^2\right)\mathbf{i} + \left((R\cos\theta)\ddot{\theta} - (R\sin\theta)\dot{\theta}^2\right)\mathbf{j}\\ &= R\ddot{\theta}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) - R\dot{\theta}^2(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}).\end{aligned}$$

Now $\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is a unit vector in the direction of the radius through the particle; we denote it by $\mathbf{e}_r$, since it is in the direction of increasing r. (The same vector, expressed in terms of Cartesian coordinates, was introduced in Exercise 1.22.) The vector $-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$ is also a unit vector but is perpendicular to $\mathbf{e}_r$ and is therefore tangential to the circle – notice that the velocity points in the direction of this vector. We denote this by $\mathbf{e}_\theta$, thinking of it as pointing in the direction of increasing θ.

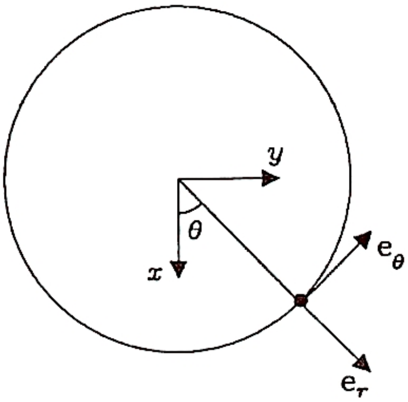

Figure 1.15

Now

$$\begin{aligned}\dot{\mathbf{r}} &= R\dot{\theta}\,\mathbf{e}_\theta,\\ \ddot{\mathbf{r}} &= R\ddot{\theta}\,\mathbf{e}_\theta - R\dot{\theta}^2\,\mathbf{e}_r.\end{aligned}$$

The constraint force has magnitude T and points towards the centre of the circle, so the corresponding force vector is $-T\mathbf{e}_r$. Thus Newton's second law gives

$$m\ddot{\mathbf{r}} = mR\ddot{\theta}\,\mathbf{e}_\theta - mR\dot{\theta}^2\,\mathbf{e}_r = \mathbf{F} - T\mathbf{e}_r.$$

This vector equation may be expressed as two scalar equations by using the fact that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are mutually orthogonal unit vectors; taking the dot product of the vector equation of motion with each in turn gives

$$-mR\dot{\theta}^2 = \mathbf{F}\cdot\mathbf{e}_r - T,$$
$$mR\ddot{\theta} = \mathbf{F}\cdot\mathbf{e}_\theta.$$

Suppose now that the total non-constraint force $\mathbf{F}$ is a function of position. The term $\mathbf{F}\cdot\mathbf{e}_\theta$ is simply the component of this force tangential to the circle; in the equations above, the force $\mathbf{F}$ must be evaluated at the particle's position on the circle, and therefore $\mathbf{F}\cdot\mathbf{e}_\theta$ may be regarded as a function of the single variable θ. A function W of θ can therefore be found such that

$$\frac{dW}{d\theta} = -\mathbf{F}\cdot\mathbf{e}_\theta.$$

(The function W is not actually a potential energy function for the system, though it plays a similar role; this is because θ is not a length coordinate. Hence the notational change. We shall touch on this point again later.) The equation

$$mR\ddot{\theta} = -\frac{dW}{d\theta}$$

then serves to determine the motion of the particle around the circle. The equation $-mR\dot{\theta}^2 = \mathbf{F}\cdot\mathbf{e}_r - T$ can be used to find the magnitude of the constraint force if it should be required. In practice, one needs to use this equation only if the problem involves some kind of test of the magnitude of the constraint force, such as whether the string breaks or whether the 'hot-wheels' car leaves the track.

Exercise 1.67

For motion on a vertical circle under gravity (pendulum, bead or 'hot-wheels' car) the non-constraint force is given by $\mathbf{F} = mg\,\mathbf{i}$. Find W and write down the equation of motion (for $\ddot{\theta}$).

Exercise 1.68

For W to be properly defined as a function on the circle it must be periodic, which is to say that it should satisfy

$$W(\theta + 2\pi) = W(\theta), \qquad \text{for all } \theta.$$

The force component $\mathbf{F}\cdot\mathbf{e}_\theta$ is bound to be periodic; however, this does not imply that W, which is obtained by integration, is necessarily periodic. For example, if the force component takes the form $k\sin\theta$ (where k is constant), then W can be taken to be $k\cos\theta$, which is periodic. But if the force component were constant, say k, then W would have to be $-k\theta + l$ (where l is another constant), which is not periodic. What condition must the non-constraint force $\mathbf{F}$ satisfy in order that W shall be well defined?

A result equivalent to the conservation of energy may be obtained by the standard trick of multiplying the equation of motion (for $\ddot{\theta}$) through by $\dot{\theta}$, giving

$$mR\dot{\theta}\ddot{\theta} + \frac{dW}{d\theta}\dot{\theta} = \frac{d}{dt}\left(\tfrac{1}{2}mR\dot{\theta}^2 + W\right) = 0.$$

Thus the quantity $\frac{1}{2}mR\dot{\theta}^2 + W$ is conserved in circular motion of the type under consideration. However, this is not quite the energy; the particle's speed is $R|\dot{\theta}|$, so its kinetic energy is $\frac{1}{2}mR^2\dot{\theta}^2$, and its total energy is

$$E = \tfrac{1}{2}mR^2\dot{\theta}^2 + RW.$$

The function RW is a potential energy function.

1.6.5 Angular momentum

Another method of deriving the equation of circular motion is to use the torque law for a single particle. This course has little use for the torque law, but it is worth reminding you about it because the associated notion of angular momentum will crop up occasionally.

The angular momentum of a particle of mass m with respect to an origin O is the vector

$$\mathbf{L} = \mathbf{r} \times m\frac{d\mathbf{r}}{dt} = \mathbf{r} \times \mathbf{p}, \tag{1.50}$$

where $\mathbf{r}$ is the particle's position vector relative to O and $\mathbf{p}$ is its momentum. The torque acting on the particle is

$$\gamma = \mathbf{r} \times \mathbf{F}^{\text{tot}}, \tag{1.51}$$

where $\mathbf{F}^{\text{tot}}$ is the total force acting on it. The torque law, which is a direct consequence of Newton's equations of motion, says that

$$\frac{d\mathbf{L}}{dt} = \gamma. \tag{1.52}$$

In the case of the particle moving on a circle, the angular momentum is given by

$$\mathbf{L} = R\mathbf{e}_r \times mR\frac{d\theta}{dt}\mathbf{e}_\theta = mR^2\frac{d\theta}{dt}\mathbf{n},$$

where $\mathbf{n}$ is a unit vector perpendicular to the plane of the circle. The constraint force $-T\mathbf{e}_r$ does not contribute to the torque, since its direction is opposite to that of $\mathbf{r}$; thus

$$\gamma = R\mathbf{e}_r \times (\mathbf{F} - T\mathbf{e}_r) = R\mathbf{e}_r \times \Big((\mathbf{F}\cdot\mathbf{e}_r)\,\mathbf{e}_r + (\mathbf{F}\cdot\mathbf{e}_\theta)\,\mathbf{e}_\theta\Big) = R(\mathbf{F}\cdot\mathbf{e}_\theta)\mathbf{n}.$$

The torque law gives

$$mR^2\ddot{\theta} = R(\mathbf{F}\cdot\mathbf{e}_\theta) = -R\frac{dW}{d\theta}.$$

Thus the torque law singles out the equation of motion involving $\ddot{\theta}$, and gives it with the extra factor R required to produce the correct expression for the energy.

The quantity mR^2 is called the moment of inertia of the particle, and is usually denoted by I. It plays a role for circular motion analogous to the role of the mass in straight-line motion. The equation of motion may be written

$$I\ddot{\theta} = -\frac{dV}{d\theta},$$

where $V = RW$ is the potential energy.

For circular motion, or more generally for motion in a plane, the angular momentum vector lies in a fixed direction, normal to the plane, and so we restrict our attention to this single component which is, of course, a scalar quantity. For circular motion the angular momentum relative to the centre of the circle is $mR^2\dot{\theta} = I\dot{\theta}$. Note that if $p = I\dot{\theta}$, then the equation of motion, for motion in a circle, may be replaced by a pair of simultaneous first-order equations, namely

$$\dot{\theta} = \frac{p}{I}, \qquad \dot{p} = -\frac{dV}{d\theta},$$

in close analogy to the equations that were derived earlier for straight-line motion. Here θ plays the role of the coordinate specifying the particle's position, and corresponds to x in straight-line motion; the angular momentum takes over the role of the linear momentum in straight-line motion; and the moment of inertia replaces the mass.

Exercise 1.69

Write down the particle's total energy in terms of p and V.

1.6.6 Dimensions

In *MST204* the physical quantities which occur are expressed in terms of the appropriate SI units, such as the kilogram, the metre, the newton, and so on. There will be no need to be so careful in this course, since actual magnitudes are rarely important. However, the use of units serves an additional purpose to that of ensuring that measurements are made consistently and comparably: it also provides a check on the correctness of expressions and equations. For example, an equation which purported to equate a quantity whose units were newton metres to one whose units were $\mathrm{kg^2\,m\,s^{-1}}$ could not be correct, because 'newton' stands for $\mathrm{kg\,m\,s^{-2}}$ and so 'newton metre' means $\mathrm{kg\,m^2\,s^{-2}}$. The units of quantities on either side of an equation must agree.

This use of units has some advantages in this course, so it is worthwhile getting to know about it. If you have studied science courses you are probably familiar with it already.

In fact, as must be obvious, it does not matter which particular system of units one is using if one merely wants to check the correctness of definitions and equations: imperial units (inches and pounds) would serve just as well as SI units. The method really depends only on checking how the three basic dimensions of mass, length and time are involved in the definition of the quantities concerned. The method is therefore referred to as the method of dimensions, or ***dimensional analysis***. We use a notation for the dimensions of quantities which uses 'powers', like the notation for units, but which uses the letters M, L and T, placed in square brackets, to stand for the basic dimensions of mass, length and time. Thus to say that a quantity is a length we say that its dimensions are those of [L]. Such a quantity must be measured in units appropriate to the measurement of length, such as metres or feet – but it is not specified which of the many length units should be used.

Dimensional analysis

Derived dimensions are then written in the form $[\mathrm{M}^a\mathrm{L}^b\mathrm{T}^c]$; so that, for example, the dimensions of force are $[\mathrm{MLT}^{-2}]$, while those of speed are $[\mathrm{LT}^{-1}]$. (Just as in ordinary algebra, one does not bother to write explicitly the power 1, and one simply ignores M, L or T if its power is 0.)

Exercise 1.70

Write down the dimensions of

(i) acceleration,
(ii) linear momentum,
(iii) energy,
(iv) moment of inertia,
(v) angular momentum,
(vi) angle,
(vii) angular speed.

Example 1.15

According to Newton's law of gravitation, the gravitational force between two particles, of masses m_1 and m_2, separated by a distance r, is Gm_1m_2/r^2. What are the dimensions of the constant of proportionality G (the universal constant of gravitation)?

Solution

Suppose that G has dimensions $[\mathrm{M}^a\mathrm{L}^b\mathrm{T}^c]$; then Gm_1m_2/r^2 has dimensions $[\mathrm{M}^{a+2}\mathrm{L}^{b-2}\mathrm{T}^c]$, and is a force, whose dimensions are $[\mathrm{MLT}^{-2}]$. So $a = -1$, $b = 3$ and $c = -2$, and the dimensions of G are $[\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}]$. (It is not really necessary to be quite so long-winded about solving problems of this type – it is usually fairly easy just to read off the correct dimensions.) ☐

Exercise 1.71

In the previous section we introduced a function W by

$$\frac{dW}{d\theta} = -\mathbf{F} \cdot \mathbf{e}_\theta .$$

Given that the quantity on the right-hand side of this equation has the dimensions of force, and that θ is an angle, show that W cannot be an energy. Show that, on the other hand, RW, where R is a length, does have the correct dimensions to be an energy.

Exercise 1.72

MST204 Unit 30, Section 1

According to Kepler's third law of planetary motion, the period T of orbital rotation of a planet about the Sun is related to the semi-major axis of its orbit a by the formula $T^2 = ka^3$, where k is a constant. What are the dimensions of k?

Exercise 1.73

Unit 30, Section 1

As you may remember from *MST204*, the escape velocity v_{esc} at a point on the surface of a spherical body of mass M and radius R is a certain power of the quantity $2GM/R$, where G is the universal constant of gravitation. Is

$$v_{\text{esc}} = \frac{2GM}{R}$$

the correct formula? If not, what is the correct formula?

Solutions to the Exercises in Unit 1

Solution 1.1

Since $\cosh x = \frac{1}{2}(e^x + e^{-x})$,

$$\frac{d}{dx}\cosh x = \frac{1}{2}\frac{d}{dx}(e^x + e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \sinh x.$$

Likewise,

$$\frac{d}{dx}\sinh x = \frac{1}{2}\frac{d}{dx}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$$

Solution 1.2

First show that the derivative of $\cosh^2 x - \sinh^2 x$ is zero:

$$\frac{d}{dx}(\cosh^2 x - \sinh^2 x) = 2\cosh x \sinh x - 2\sinh x \cosh x = 0.$$

Thus $\cosh^2 x - \sinh^2 x$ is constant, and is equal to its value at any convenient value of x, such as $x = 0$. Clearly, from the definitions, $\cosh 0 = 1$ and $\sinh 0 = 0$; thus

$$\cosh^2 x - \sinh^2 x = 1.$$

Solution 1.3

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) = -\frac{\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$$

$$\frac{d}{dx}\operatorname{cosech} x = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = -\frac{\cosh x}{\sinh^2 x} = -\coth x \operatorname{cosech} x$$

$$\frac{d}{dx}\coth x = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\frac{1}{\sinh^2 x} = -\operatorname{cosech}^2 x$$

Solution 1.4

Note that $\cosh x$ is always positive; thus the derivative of $\sinh x$ is everywhere positive, which means that $\sinh x$ is increasing. Furthermore,

$$\sinh(-x) = \frac{1}{2}(e^{-x} - e^x) = -\sinh x,$$

so $\sinh x$ is odd. Thus, in particular, the equation $\sinh x = 0$ has precisely one root, at $x = 0$. It follows that the derivative of $\cosh x$ is zero just once, at $x = 0$, and is positive for $x > 0$ and negative for $x < 0$; thus $\cosh x$ has precisely one stationary point, a minimum at $x = 0$. Finally

$$\cosh(-x) = \frac{1}{2}(e^{-x} + e^x) = \cosh x,$$

so $\cosh x$ is even. The graphs of the two functions are shown overleaf.

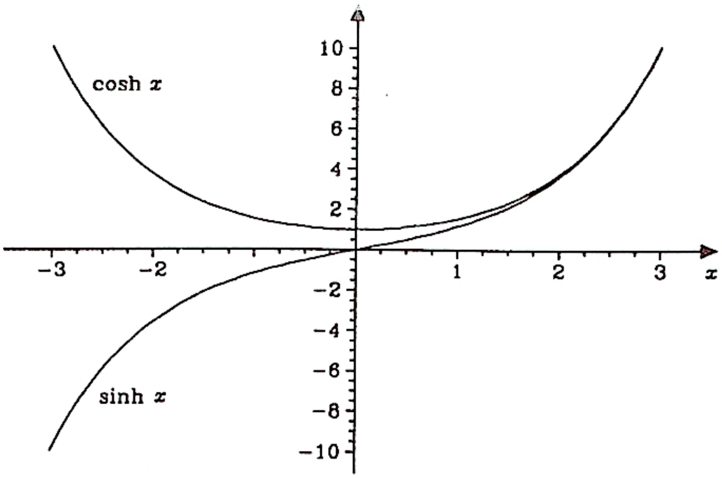

Figure S1.1

Solution 1.5

Since $\sinh x$ is odd and $\cosh x$ is even, $\tanh x$ is odd; moreover, its derivative, $\operatorname{sech}^2 x$, is always positive and so $\tanh x$ is increasing. However, unlike $\sinh x$, $\tanh x$ is bounded:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} \to 1 \qquad \text{as} \quad x \to +\infty;$$

and similarly

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \to -1 \qquad \text{as} \quad x \to -\infty.$$

The graph is shown below.

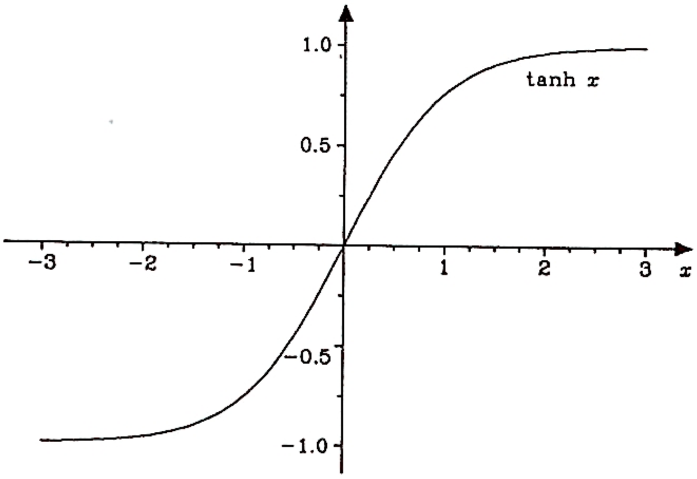

Figure S1.2

Solution 1.6

If $y = \sinh x$, the graph in Figure S1.1 shows that for every value of y there is only one value of x, that is, $x(y)$ is single-valued. The equation may be written in the form

$$y = \tfrac{1}{2}(e^x - e^{-x}) \qquad \text{or} \qquad 2ye^x = e^{2x} - 1.$$

This can be rewritten as a quadratic in e^x,

$$e^{2x} - 2ye^x - 1 = 0,$$

having the solutions

$$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}.$$

However, e^x must be positive, which rules out the negative sign in front of the square root, leaving

$$x = \ln(y + \sqrt{y^2+1}).$$

Note that there is exactly one solution, which is consistent with the fact that the function $\sinh x$ is increasing.

For $\cosh x$, we have to be more careful as x is defined only for $y \geq 1$, and when $y > 1$ there are two values of x which satisfy the equation. The same method as above gives

$$y = \tfrac{1}{2}(e^x + e^{-x}) \qquad \text{or} \qquad 2ye^x = e^{2x} + 1,$$

again giving a quadratic in e^x,

$$e^{2x} - 2ye^x + 1 = 0,$$

having the solutions

$$e^x = \frac{2y \pm \sqrt{4y^2-4}}{2} = y \pm \sqrt{y^2-1},$$

so there are in general two roots, as expected.

In order for the roots to be real, y must satisfy $|y| \geq 1$. For $y > 1$ both roots are positive, giving the two solutions

$$x = \ln(y \pm \sqrt{y^2-1}),$$

but since $y - \sqrt{y^2-1} = 1/(y + \sqrt{y^2-1})$ we have

$$\ln(y - \sqrt{y^2-1}) = -\ln(y + \sqrt{y^2-1}),$$

so the two roots are

$$x = \pm\ln(y + \sqrt{y^2-1}).$$

For $y = 1$ there is the single solution $x = 0$.

For $y \leq -1$ the expression $y \pm \sqrt{y^2-1}$ is negative, whichever sign is chosen, so there are no solutions for x.

These results are again consistent with the nature of the function $\cosh x$, as shown by its graph in Figure S1.1.

Solution 1.7

Since

$$\frac{d^2}{dx^2}\cosh x = \frac{d}{dx}\sinh x = \cosh x$$

and

$$\frac{d^2}{dx^2}\sinh x = \frac{d}{dx}\cosh x = \sinh x,$$

both $\cosh x$ and $\sinh x$ satisfy the differential equation

$$\frac{d^2y}{dx^2} - y = 0.$$

Since this is a linear equation, the linear combination $A\cosh x + B\sinh x$ is also a solution; since the equation is second-order and there are two arbitrary constants, this is the general solution.

On the other hand,

$$\frac{d^2}{dx^2}\cosh(x+a) = \frac{d}{dx}\sinh(x+a) = \cosh(x+a),$$

and so $\cosh(x+a)$ is also a solution. It follows that

$$\cosh(x+a) = A\cosh x + B\sinh x$$

for suitable constants A and B. To evaluate these constants, use initial conditions: when $x = 0$,

$$\cosh a = A,$$

while differentiating and setting $x = 0$ gives

$$\sinh a = B.$$

Thus

$$\cosh(x+a) = \cosh x\cosh a + \sinh x\sinh a.$$

The function $\sinh(x+a)$ is yet another solution of the same differential equation; thus

$$\sinh(x+a) = C\cosh x + D\sinh x$$

for suitable C and D, and

$$\sinh a = C, \qquad \cosh a = D.$$

Thus

$$\sinh(x+a) = \cosh x\sinh a + \sinh x\cosh a.$$

On setting $a = x$, one obtains the formulae

$$\begin{aligned}\cosh 2x &= \cosh^2 x + \sinh^2 x,\\ \sinh 2x &= 2\cosh x\sinh x.\end{aligned}$$

Solution 1.8

The higher derivatives of $\cosh x$ are easy to compute:

$$\frac{d^{2n}}{dx^{2n}}\cosh x = \cosh x, \qquad \frac{d^{2n+1}}{dx^{2n+1}}\cosh x = \sinh x, \qquad n = 0, 1, 2, \ldots.$$

Thus

$$\frac{d^{2n}}{dx^{2n}}\cosh x|_{x=0} = 1, \qquad \frac{d^{2n+1}}{dx^{2n+1}}\cosh x|_{x=0} = 0, \qquad n = 0, 1, 2, \ldots$$

and the Taylor series is

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots.$$

For $\sinh x$, on the other hand,

$$\frac{d^{2n}}{dx^{2n}}\sinh x = \sinh x, \qquad \frac{d^{2n+1}}{dx^{2n+1}}\sinh x = \cosh x, \qquad n = 0, 1, 2, \ldots.$$

Thus

$$\frac{d^{2n}}{dx^{2n}}\sinh x|_{x=0} = 0, \qquad \frac{d^{2n+1}}{dx^{2n+1}}\sinh x|_{x=0} = 1, \qquad n = 0, 1, 2, \ldots$$

and the Taylor series for $\sinh x$ is

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots.$$

Solution 1.9

(i) Since $\sinh x = \dfrac{d}{dx}\cosh x$, the integral can be written as

$$\int dx\, \sinh x = \int dx\, \frac{d}{dx}\cosh x = \cosh x.$$

(ii) Similarly, $\cosh x = \dfrac{d}{dx}\sinh x$, so

$$\int dx\, \cosh x = \int dx\, \frac{d}{dx}\sinh x = \sinh x.$$

(iii) Since $\cosh^2 x - \sinh^2 x = 1$, the form of the integrand suggests that for $x > 0$ we make the substitution $x = a\cosh y$, to give

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int dy\, \frac{a\sinh y}{a\sinh y} = y = \cosh^{-1}(x/a), \qquad \text{for} \quad x > 0.$$

If $x < 0$ then we must put $x = -a\cosh y$ to give $-\cosh^{-1}(-x/a)$.

(iv) Again the form of the integrand suggests defining $x = a\cosh y$, if $x > 0$, to give

$$\int dx\, \sqrt{x^2 - a^2} = a^2 \int dy\, \sinh^2 y.$$

Now use the result $\cosh 2y = \cosh^2 y + \sinh^2 y = 1 + 2\sinh^2 y$ of Exercise 1.7 to rewrite the integrand as

$$\begin{aligned}\int dx\, \sqrt{x^2 - a^2} &= \frac{a^2}{2}\int dy\, (\cosh 2y - 1)\\ &= \frac{a^2}{4}(\sinh 2y - 2y)\\ &= \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}(x/a), \qquad \text{for} \quad x > 0.\end{aligned}$$

(v) Use the substitution $x = a\sinh y$, which is suggested by the fact that $\sinh^2 y + 1 = \cosh^2 y$. So $dx = a\cosh y\, dy$ and

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + a^2}} &= \int dy\, \frac{a\cosh y}{\sqrt{a^2\sinh^2 y + a^2}}\\ &= \int dy\, \frac{\cosh y}{\cosh y}\\ &= \int dy = y = \sinh^{-1}(x/a).\end{aligned}$$

(vi) Use the substitution $y = \cosh x$, suggested by the fact that $\tanh x$ is a quotient involving this function and its derivative; then

$$\int dx\, \tanh x = \int \frac{dy}{y} = \ln y = \ln\cosh x.$$

This is an example of the general rule that

$$\int dx\, \frac{f'(x)}{f(x)} = \ln(f(x)), \qquad \text{if} \quad f(x) > 0.$$

Solution 1.10

From the equation

$$\frac{\partial f}{\partial x} = g(y)$$

it follows (by integrating with respect to x, treating y as a constant) that

$$f(x, y) = g(y)x + a(y),$$

where $a(y)$ is an undetermined function of y. But

$$\begin{aligned}\frac{\partial f}{\partial y} &= h(x) = kx + m\\ &= \frac{dg}{dy}x + \frac{da}{dy},\end{aligned}$$

whence (equating the coefficients of x and the constant terms)

$$\frac{dg}{dy} = k, \qquad \frac{da}{dy} = m.$$

It follows that $g(y) = ky + l$, as is known; and that

$$a(y) = my + n,$$

so that

$$f(x, y) = kxy + lx + my + n.$$

Solution 1.11

If there were such a function, then

$$\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = 1$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) = -1.$$

But for a well-behaved function $h(x, y)$ these should be equal; and since they are clearly not equal, the conclusion is that there can be no well-behaved function satisfying these conditions.

Solution 1.12

This integral can be evaluated by integrating by parts twice, but it is easier to notice that

$$x^2 \sin xy = -\frac{\partial^2}{\partial y^2} \sin xy,$$

so that

$$\begin{aligned}f(y) &= -\int_0^\pi dx\, \frac{\partial^2}{\partial y^2} \sin xy = -\frac{d^2}{dy^2}\int_0^\pi dx \sin xy\\ &= -\frac{d^2}{dy^2}\left[-\frac{1}{y}\cos xy\right]_0^\pi = \frac{d^2}{dy^2}\left(\frac{\cos \pi y - 1}{y}\right)\\ &= \frac{2(\cos \pi y - 1)}{y^3} + \frac{2\pi \sin \pi y}{y^2} - \frac{\pi^2 \cos \pi y}{y}.\end{aligned}$$

Notice that this method has replaced a double integration by parts by a second derivative, which is generally easier to evaluate.

Solution 1.13

$$\operatorname{grad} h = 2(ax + by)\mathbf{i} + 2(bx + cy)\mathbf{j}$$

Solution 1.14

Suppose that $u(x,y)\,\mathbf{i}+v(x,y)\,\mathbf{j}$ is a gradient. Then there is some function $f(x,y)$ such that

$$u(x,y)=\frac{\partial f}{\partial x}(x,y), \qquad v(x,y)=\frac{\partial f}{\partial y}(x,y).$$

But then

$$\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial u}{\partial y}.$$

Solution 1.15

The line is a contour of the function $h(x,y)=3x-4y$. The gradient of the function will be a vector (field) perpendicular to the contours of the function. Now

$$\operatorname{grad} h = 3\,\mathbf{i}-4\,\mathbf{j}.$$

A unit vector perpendicular to the line is obtained by finding a unit vector in the direction of $\operatorname{grad} h$, which is computed by dividing $\operatorname{grad} h$ by its magnitude, $\sqrt{3^2+4^2}=5$:

$$\pm\left(\tfrac{3}{5}\,\mathbf{i}-\tfrac{4}{5}\,\mathbf{j}\right).$$

Note the two possible vectors.

Solution 1.16

The first-order Taylor approximation gives

$$\begin{aligned}F(t)&=f(a,b)+t\left(\frac{\partial f}{\partial x}(a,b)\xi+\frac{\partial f}{\partial y}(a,b)\eta\right)+O(t^2)\\&=F(0)+t\,\mathbf{e}\cdot\operatorname{grad} f+O(t^2),\end{aligned}$$

and so

$$\frac{F(t)-F(0)}{t}=\mathbf{e}\cdot\operatorname{grad} f+O(t).$$

Hence, on taking the limit as $t\to 0$,

$$\frac{dF}{dt}(0)=\mathbf{e}\cdot\operatorname{grad} f.$$

Solution 1.17

Using the Taylor approximation of order two for a function of one variable and treating y as constant gives

$$f(x,y)=f(a,y)+(x-a)\frac{\partial f}{\partial x}(a,y)+\tfrac{1}{2}(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,y)+O((x-a)^3).$$

Now Taylor expand the terms on the right-hand side in powers of $(y-b)$, as far as is necessary to obtain the terms jointly of second order in $(x-a)$ and $(y-b)$:

$$\begin{aligned}f(a,y)&=f(a,b)+(y-b)\frac{\partial f}{\partial y}(a,b)+\tfrac{1}{2}(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)+O((y-b)^3),\\ \frac{\partial f}{\partial x}(a,y)&=\frac{\partial f}{\partial x}(a,b)+(y-b)\frac{\partial^2 f}{\partial x\partial y}(a,b)+O((y-b)^2),\\ \frac{\partial^2 f}{\partial x^2}(a,y)&=\frac{\partial^2 f}{\partial x^2}(a,b)+O(y-b);\end{aligned}$$

substituting these into the previous equation gives

$$\begin{aligned}f(x,y)=f(a,b)&+(x-a)\frac{\partial f}{\partial x}(a,b)+(y-b)\frac{\partial f}{\partial y}(a,b)\\&+\tfrac{1}{2}(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b)+(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}(a,b)\\&+\tfrac{1}{2}(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)+O(|\mathbf{r}-\mathbf{a}|^3).\end{aligned}$$

See the text following Equation 1.7, page 14, for an explanation of the notation $O(|\mathbf{r}-\mathbf{a}|^3)$.

Solution 1.18

By direct evaluation,

$$\begin{aligned} h(t) &= (2t)^2 + (2t)(1-3t) - (1-3t)^2 \\ &= 4t^2 + 2t - 6t^2 - 1 + 6t - 9t^2 \\ &= -1 + 8t - 11t^2; \end{aligned}$$

thus

$$\frac{dh}{dt} = 8 - 22t.$$

Evaluating using the chain rule,

$$\begin{aligned} &\frac{\partial h}{\partial x}(x,y) = 2x + y, \qquad \frac{\partial h}{\partial y}(x,y) = x - 2y, \\ &\frac{\partial h}{\partial x}(x(t), y(t)) = 2(2t) + (1-3t) = 1 + t, \\ &\frac{\partial h}{\partial y}(x(t), y(t)) = (2t) - 2(1-3t) = -2 + 8t, \\ &\frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = -3. \end{aligned}$$

Thus

$$\begin{aligned} \frac{dh}{dt} &= \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} \\ &= 2(1+t) - 3(-2+8t) = 8 - 22t, \end{aligned}$$

as before.

Solution 1.19

The interpretation is exactly the same as before: the rate of change of the scalar field h along a curve can be computed by taking the dot product of the gradient of h with the tangent vector to the curve. But now h is a scalar field and the curve is a curve in three-dimensional space.

Solution 1.20

The required formula is

$$\frac{dh}{dz} = \frac{\partial h}{\partial x}\frac{dx}{dz} + \frac{\partial h}{\partial y}\frac{dy}{dz} + \frac{\partial h}{\partial z}.$$

This may be inferred from the chain rule for functions of three variables by thinking of z as being given in terms of itself by the identity function, whose derivative is just 1; in other words, by writing

$$\frac{dh}{dz} = \frac{\partial h}{\partial x}\frac{dx}{dz} + \frac{\partial h}{\partial y}\frac{dy}{dz} + \frac{\partial h}{\partial z}\frac{dz}{dz}$$

and setting $dz/dz = 1$.

Solution 1.21

$$\begin{aligned} \frac{dh}{dt} &= \frac{\partial h}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial h}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial h}{\partial x_n}\frac{dx_n}{dt} \\ &= \sum_{k=1}^{n} \frac{\partial h}{\partial x_k}\frac{dx_k}{dt} \end{aligned}$$

Solution 1.22

First compute the partial derivatives of r:

$$2r\frac{\partial r}{\partial x} = 2x, \qquad \text{so} \quad \frac{\partial r}{\partial x} = \frac{x}{r};$$

and similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

Thus

$$\begin{aligned}\operatorname{grad} h &= \frac{\partial h}{\partial x}\mathbf{i} + \frac{\partial h}{\partial y}\mathbf{j}\\ &= \frac{df}{dr}\frac{\partial r}{\partial x}\mathbf{i} + \frac{df}{dr}\frac{\partial r}{\partial y}\mathbf{j}\\ &= \frac{df}{dr}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j}\right)\\ &= \frac{df}{dr}\mathbf{e}_r.\end{aligned}$$

Solution 1.23

$$\frac{\partial h}{\partial y} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial y}$$

Solution 1.24

$$h(u,v) = u\cos v, \qquad u(x,y) = \sqrt{x^2+y^2}, \qquad v(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

give

$$\begin{aligned}\frac{\partial h}{\partial u} &= \cos v = \frac{x}{\sqrt{x^2+y^2}},\\ \frac{\partial h}{\partial v} &= -u\sin v = -y,\end{aligned}$$

as in the text; while

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}},\\ \frac{\partial v}{\partial y} &= \frac{1}{1+(y/x)^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}.\end{aligned}$$

Thus

$$\frac{\partial h}{\partial y} = \left(\frac{x}{\sqrt{x^2+y^2}}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + (-y)\left(\frac{x}{x^2+y^2}\right) = 0.$$

Solution 1.25

$$\begin{aligned}\frac{\partial h}{\partial x} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial h}{\partial w}\frac{\partial w}{\partial x},\\ \frac{\partial h}{\partial y} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial h}{\partial w}\frac{\partial w}{\partial y}.\end{aligned}$$

Solution 1.26

$$\begin{aligned}\frac{\partial h}{\partial x_1} &= \frac{\partial h}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial h}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \cdots + \frac{\partial h}{\partial u_m}\frac{\partial u_m}{\partial x_1},\\ \frac{\partial h}{\partial x_2} &= \frac{\partial h}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial h}{\partial u_2}\frac{\partial u_2}{\partial x_2} + \cdots + \frac{\partial h}{\partial u_m}\frac{\partial u_m}{\partial x_2},\\ &\vdots\\ \frac{\partial h}{\partial x_n} &= \frac{\partial h}{\partial u_1}\frac{\partial u_1}{\partial x_n} + \frac{\partial h}{\partial u_2}\frac{\partial u_2}{\partial x_n} + \cdots + \frac{\partial h}{\partial u_m}\frac{\partial u_m}{\partial x_n}.\end{aligned}$$

These formulae may be written more economically by using the summation symbol:

$$\frac{\partial h}{\partial x_i} = \sum_{k=1}^{m} \frac{\partial h}{\partial u_k}\frac{\partial u_k}{\partial x_i}, \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

Solution 1.27

(i) Linear (ii) Neither (iii) Both (iv) Separable

Solution 1.28

In order that the equation

$$\frac{dy}{dx} = m(x, y)$$

should be both linear and separable, the function $m(x, y)$ must be a linear function of y and a product of a function of x and a function of y; so it must be of the form $f(x)g(y)$, where $g(y)$ is a linear function of y, say $g(y) = ky + l$ where k and l are *constants*. Thus the equation must be of the form

$$\frac{dy}{dx} = f(x)(ky + l).$$

An alternative way of saying the same thing is that $m(x, y)$ must be linear in y, say $m(x, y) = k(x)y + l(x)$, where $l(x)$ is a *constant* multiple of $k(x)$.

Solution 1.29

Solution by separation of variables gives

$$\frac{dy}{dx} = xy \quad \Longrightarrow \quad \int \frac{dy}{y} = \int dx\, x.$$

For $y > 0$ integration gives

$$\ln y = \tfrac{1}{2}x^2 + C \qquad \text{or} \qquad y = ke^{x^2/2}, \qquad y > 0,$$

where $k = e^C$. For $y < 0$ we obtain

$$\ln(-y) = \tfrac{1}{2}x^2 + C' \qquad \text{or} \qquad y = k'e^{x^2/2}, \qquad y < 0,$$

where $k' = -e^{C'}$.

The two cases may be combined into the single formula

$$y = Ke^{x^2/2},$$

where K may be either positive or negative; and in fact K may be zero, because the constant function $y = 0$ is also a solution (by inspection), though the method of separation of variables does not produce this solution.

Now consider the solution by the integrating factor method. To compute the integrating factor, first calculate

$$\int dx\,(-x) = -\tfrac{1}{2}x^2.$$

The integrating factor is thus

$$e^{-x^2/2},$$

and the equation obtained by multiplying through by the integrating factor and rearranging terms is

$$e^{-x^2/2}\left(\frac{dy}{dx} - xy\right) = 0 \quad \Longrightarrow \quad \frac{d}{dx}\left(e^{-x^2/2}y\right) = 0,$$

so integration gives

$$e^{-x^2/2}y = K \qquad \text{or} \qquad y = Ke^{x^2/2}.$$

Note that with this method all the solutions, including $y = 0$, are produced in one go. The solution with value 0 when $x = 0$ is the constant solution $y = 0$, which strictly speaking cannot be obtained by the separation of variables method.

Solution 1.30

For the equation

$$\frac{dy}{dx} - y = e^x,$$

the general solution of the homogeneous equation,

$$\frac{dy}{dx} - y = 0,$$

is

$$y = ce^x,$$

for some constant c. The right-hand side of the inhomogeneous equation is a solution of the homogeneous equation, as noted in the text, so try the solution $y = kxe^x$. Then

$$\frac{dy}{dx} = ke^x + kxe^x \quad \Longrightarrow \quad \frac{dy}{dx} - y = ke^x,$$

so that putting $k = 1$ gives the particular solution $y_p = xe^x$. Thus the general solution is

$$y = xe^x + ce^x.$$

For the equation

$$\frac{dy}{dx} + y = e^x,$$

the general solution of the homogeneous equation,

$$\frac{dy}{dx} + y = 0,$$

is

$$y = ce^{-x},$$

for some constant c. The appropriate trial solution is now

$$y = ke^x,$$

giving

$$\frac{dy}{dx} = ke^x \quad \Longrightarrow \quad \frac{dy}{dx} + y = 2ke^x,$$

so that $k = \frac{1}{2}$ provides the particular solution. Thus the general solution is

$$y = \tfrac{1}{2}e^x + ce^{-x}.$$

Solution 1.31

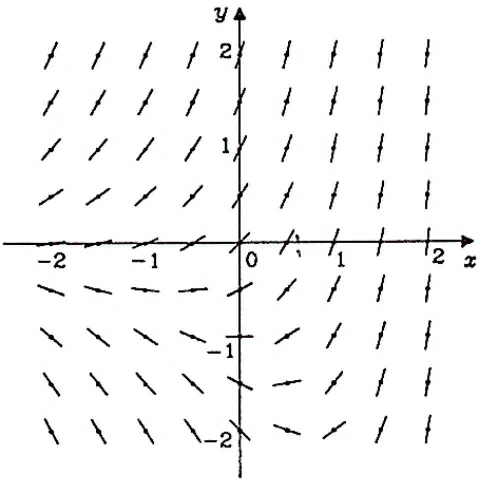

Figure S1.3

Solution 1.32

If

$$\mathbf{A}(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix},$$

then

$$\mathbf{A}(s)\mathbf{A}(t) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s+t & 1 \end{bmatrix} = \mathbf{A}(s+t);$$

while if

$$\mathbf{A}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

then

$$\begin{aligned} \mathbf{A}(s)\mathbf{A}(t) &= \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \\ &= \begin{bmatrix} \cos s \cos t - \sin s \sin t & \cos s \sin t + \sin s \cos t \\ -\sin s \cos t - \cos s \sin t & -\sin s \sin t + \cos s \cos t \end{bmatrix}. \end{aligned}$$

Now

$$\cos s \cos t - \sin s \sin t = \cos(s+t)$$

and

$$\sin s \cos t + \cos s \sin t = \sin(s+t),$$

and therefore

$$\mathbf{A}(s)\mathbf{A}(t) = \begin{bmatrix} \cos(s+t) & \sin(s+t) \\ -\sin(s+t) & \cos(s+t) \end{bmatrix} = \mathbf{A}(s+t).$$

Similarly, $\mathbf{A}(t)\mathbf{A}(s) = \mathbf{A}(t+s) = \mathbf{A}(s+t)$.

Solution 1.33

Suppose that $\mathbf{u} = [u_1 \quad u_2]^T$ and $\mathbf{v} = [v_1 \quad v_2]^T$, and that

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$\mathbf{BA} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} au_1 + bu_2 & av_1 + bv_2 \\ cu_1 + du_2 & cv_1 + dv_2 \end{bmatrix} = [\mathbf{Bu} \quad \mathbf{Bv}].$$

Solution 1.34

$$\begin{aligned} [\mathbf{u} \quad \mathbf{v}] \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} u_1 a + v_1 b \\ u_2 a + v_2 b \end{bmatrix} \\ &= a \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + b \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = a\,\mathbf{u} + b\,\mathbf{v}. \end{aligned}$$

Solution 1.35

$$\mathbf{Ai} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{u}$$

and $$\mathbf{Aj} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{v}.$$

Thus

$$\mathbf{A} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{A}(a\,\mathbf{i} + b\,\mathbf{j}) = a\,\mathbf{Ai} + b\,\mathbf{Aj} = a\,\mathbf{u} + b\,\mathbf{v}.$$

Solution 1.36

Apply the result of the previous two exercises to the columns of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in turn.

Solution 1.37

Clearly, if either $a = c = 0$, or $b = ka$ and $d = kc$, then $ad - bc = 0$. Suppose, on the other hand, that $ad - bc = 0$, and that $a \neq 0$: then k can be defined to be b/a, giving

$$d = \frac{b}{a}c = kc.$$

If $a = 0$ and $c \neq 0$, then k can be defined to be d/c; but b must be zero, and so $b = ka$ is then satisfied. Thus either $a = c = 0$ or there is a number k such that $b = ka$ and $d = kc$. But these two conditions are the two alternatives which may occur if the column vectors $[a \quad c]^T$ and $[b \quad d]^T$ are linearly dependent: for these vectors are linearly independent if and only if there are numbers k' and l', not both zero, such that

$$k' \begin{bmatrix} a \\ c \end{bmatrix} + l' \begin{bmatrix} b \\ d \end{bmatrix} = 0,$$

whence either $a = c = 0$ (if $l' = 0$) or $b = ka$ and $d = kc$ with $k = -k'/l'$.

Equally, the conditions $a = b = 0$, or $c = ka$ and $d = kb$ for some k, are necessary and sufficient for $ad - bc$ to be zero; and these are equivalent to the linear dependence of the *rows* of **A**.

Solution 1.38

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + da \end{bmatrix} = (\det \mathbf{A})\mathbf{I}$$

and

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & 0 \\ 0 & -cb + ad \end{bmatrix} = (\det \mathbf{A})\mathbf{I}.$$

Solution 1.39

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad,$$

$$\begin{vmatrix} r\cos\alpha & r\sin\alpha \\ -r\sin\alpha & r\cos\alpha \end{vmatrix} = r^2\cos^2\alpha - (-r^2\sin^2\alpha) = r^2,$$

$$\begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1.$$

The first matrix will be singular if and only if either $a = 0$ or $d = 0$ (or both); that is, if and only if it has either a row or a column of zeros. The second will be singular if and only if $r = 0$; that is, if and only if the matrix is the zero matrix. The third can never be singular.

Solution 1.40

If **A** is non-singular, with inverse $\mathbf{A}^{-1}$, then

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I},$$

and therefore by the multiplicative property of determinants

$$(\det \mathbf{A})(\det \mathbf{A}^{-1}) = \det \mathbf{I} = 1,$$

whence

$$\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1}.$$

By the same property,

$$\begin{aligned}\det(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) &= (\det \mathbf{P}^{-1})(\det \mathbf{A})(\det \mathbf{P})\\ &= (\det \mathbf{P})^{-1}(\det \mathbf{A})(\det \mathbf{P})\\ &= (\det \mathbf{P})^{-1}(\det \mathbf{P})(\det \mathbf{A}) = \det \mathbf{A}.\end{aligned}$$

Solution 1.41

To compute the area of the parallelogram whose sides are $\mathbf{Au}$ and $\mathbf{Av}$, compute the determinant of the matrix whose columns are these vectors. But this matrix is $\mathbf{A}[\mathbf{u}\quad \mathbf{v}]$, whose determinant is $(\det \mathbf{A})(\det[\mathbf{u}\quad \mathbf{v}])$. Thus the area of the parallelogram whose sides are $\mathbf{Au}$ and $\mathbf{Av}$ is $|\det \mathbf{A}|$ times the area of the parallelogram whose sides are $\mathbf{u}$ and $\mathbf{v}$.

Solution 1.42

The characteristic equation is

$$(-\lambda)(4-\lambda)-(-1)4=0 \quad\Longrightarrow\quad \lambda^2-4\lambda+4=0 \quad\Longrightarrow\quad (\lambda-2)^2=0.$$

There is only one eigenvalue, $\lambda = 2$. The corresponding eigenvectors $[u\quad v]^T$ satisfy

$$\begin{aligned}-2u-v&=0,\\ 4u+2v&=0,\end{aligned}$$

with solution

$$v=-2u.$$

The eigenvectors corresponding to the eigenvalue 2 are thus the non-zero scalar multiples of $[1\quad -2]^T$.

Solution 1.43

$$\begin{vmatrix} a-\lambda & b\\ c & d-\lambda\end{vmatrix} \begin{aligned}&= (a-\lambda)(d-\lambda)-bc\\ &= \lambda^2-(a+d)\lambda+(ad-bc)\\ &= \lambda^2-\lambda\ \mathrm{tr}\,\mathbf{A}+\det\mathbf{A}\end{aligned}$$

Solution 1.44

(i) The eigenvalues are given by the solution of the quadratic equation found in the previous question; these are

$$\lambda=\frac{\mathrm{tr}\,\mathbf{A}\pm\left((\mathrm{tr}\,\mathbf{A})^2-4\det\mathbf{A}\right)^{1/2}}{2}.$$

These are real only if

$$\Delta=(\mathrm{tr}\,\mathbf{A})^2-4\det\mathbf{A}=(a+d)^2-4ad+4bc=(a-d)^2+4bc\geq 0$$

and are distinct only if $\Delta > 0$.

(ii) The eigenvectors $[u\quad v]^T$ satisfy the equation

$$\begin{bmatrix} a-\lambda & b\\ c & d-\lambda\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=0,$$

where λ is one of the eigenvalues. In component form these are

$$\begin{aligned}(a-\lambda)u+bv&=0,\\ cu+(d-\lambda)v&=0.\end{aligned}$$

The first of these has a solution $u=b$, $v=(\lambda-a)$, giving the required two eigenvectors, $[b\quad \lambda_k-a]^T$.

Alternatively, we could use the second equation, to give the two eigenvectors $[\lambda_k-d\quad c]^T$. It is not immediately obvious but, apart from a constant, these are the same as the previous pair of eigenvectors: this follows because of the relations

$$\lambda_1 + \lambda_2 = \operatorname{tr} \mathbf{A} = a + d,$$
$$\lambda_1\lambda_2 = \det \mathbf{A} = ad - bc.$$

Consider the vector $[\lambda_1 - d \quad c]^T$ and multiply by the constant $b/(\lambda_1 - d)$. Then the second component becomes

$$\frac{bc}{\lambda_1 - d} = \frac{ad - \lambda_1\lambda_2}{a - \lambda_2} = \frac{a(\lambda_1 + \lambda_2 - a) - \lambda_1\lambda_2}{a - \lambda_2} = \frac{(\lambda_1 - a)(a - \lambda_2)}{a - \lambda_2},$$

where we have used the relations between the sum and product of the eigenvalues given above. Thus

$$\frac{b}{\lambda_1 - d}[\lambda_1 - d \quad c]^T = [b \quad \lambda_1 - a]^T.$$

Solution 1.45

The characteristic equation of the matrix

$$\mathbf{A}_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

is $\qquad (x - \lambda)^2 = 0$

(using x for the unknown eigenvalue), which has the single root $x = \lambda$. The equations for the corresponding eigenvectors vanish identically; so every non-zero vector is an eigenvector, and in particular the two linearly independent vectors $[1 \quad 0]^T$ and $[0 \quad 1]^T$ are eigenvectors. In fact, these results are quite obvious from a cursory inspection of the matrix itself: $\mathbf{A}_1$ is a multiple of the unit matrix, and its action on any vector $\mathbf{u}$ is simply

$$\mathbf{A}_1\mathbf{u} = \lambda\mathbf{u};$$

thus clearly λ is an eigenvalue of $\mathbf{A}_1$, with every non-zero vector as a corresponding eigenvector, and $\mathbf{A}_1$ can have no other eigenvalue.

The characteristic equation for

$$\mathbf{A}_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

is likewise

$$(x - \lambda)^2 = 0$$

with the single root λ as before. But now an eigenvector $[u \quad v]^T$ must satisfy

$$u = 0;$$

so the eigenvectors are $[0 \quad 1]^T$ and its non-zero scalar multiples.

Solution 1.46

The condition for the quadratic equation

$$x^2 - \beta x + \gamma = 0$$

to have coincident roots is

$$\beta^2 - 4\gamma = 0,$$

and when this condition is satisfied, the common value of the roots is just $\frac{1}{2}\beta$. Thus the characteristic equation

$$\lambda^2 - \lambda \operatorname{tr} \mathbf{A} + \det \mathbf{A} = 0$$

has coincident roots if and only if

$$\Delta = (\operatorname{tr} \mathbf{A})^2 - 4 \det \mathbf{A} = (a - d)^2 + 4bc = 0.$$

The common value of the roots is then $\frac{1}{2}\operatorname{tr}\mathbf{A}$. The eigenvector equations are

$$au + bv = \tfrac{1}{2}(a+d)u,$$
$$cu + dv = \tfrac{1}{2}(a+d)v,$$

or equivalently

$$(a-d)u + 2bv = 0,$$
$$2cu - (a-d)v = 0.$$

The determinant of the matrix of coefficients of these equations is

$$-(a-d)^2 - 4bc = 0,$$

so the equations are not independent. For there to be two linearly independent solutions of these equations, every vector $[u \quad v]^T$ must satisfy them (since they are linear equations), which means that their coefficients must all vanish; in this case $a = d$ and $b = c = 0$, so the matrix is a multiple of the unit matrix.

Solution 1.47

(i) To find the conditions satisfied by $\mathbf{k}$ and $\mathbf{u}$ in the general case, merely write down the conditions satisfied by the corresponding vectors in the case in which $\operatorname{tr}\mathbf{A} = 0$ and replace $\mathbf{A}$ by $\mathbf{A} - \lambda\mathbf{I}$. This procedure gives

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = 0,$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{k} = \mathbf{u},$$

or in other words

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u},$$
$$\mathbf{A}\mathbf{k} = \mathbf{u} + \lambda\mathbf{k}.$$

(ii) The equation for the eigenvectors $[u \quad v]^T$ is

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

and as in Solution 1.45, a solution is $\mathbf{u} = [b \quad a-\lambda]^T$. But $\lambda = \frac{1}{2}(a+d)$, so that $\mathbf{u} = [b \quad \frac{1}{2}(d-a)]^T$.

The equation for $\mathbf{k}$ is

$$\begin{bmatrix} \frac{1}{2}(a-d) & b \\ c & -\frac{1}{2}(a-d) \end{bmatrix}\mathbf{k} = \mathbf{u}.$$

The vector $[0 \quad 1]^T$ is easily seen to be a solution of this equation.

Solution 1.48

The characteristic equation of the matrix

$$\begin{bmatrix} r\cos\alpha & r\sin\alpha \\ -r\sin\alpha & r\cos\alpha \end{bmatrix}$$

is $\qquad \lambda^2 - 2r\lambda\cos\alpha + r^2 = 0,$

whose roots are

$$\lambda = r(\cos\alpha \pm i\sin\alpha) = re^{\pm i\alpha}.$$

In order that $[u \quad v]^T$ be an eigenvector corresponding to the eigenvalue $r(\cos\alpha + i\sin\alpha)$, its components must satisfy

$$-iru\sin\alpha + rv\sin\alpha = 0,$$
$$-ru\sin\alpha - irv\sin\alpha = 0.$$

The second of these equations is just $-i$ times the first. Thus $[1 \quad i]^T$ is an eigenvector corresponding to the eigenvalue $r(\cos\alpha + i\sin\alpha)$, and so is any *complex* scalar multiple of this one. Likewise, $[1 \quad -i]^T$ is an eigenvector corresponding to the eigenvalue $r(\cos\alpha - i\sin\alpha)$, and so is any complex scalar multiple of it, e.g. $[i \quad 1]^T$.

Solution 1.49

$$\mathbf{A}(\mathbf{u} + i\,\mathbf{v}) = (\mu + i\nu)(\mathbf{u} + i\,\mathbf{v})$$

gives

$$\mathbf{A}\mathbf{u} + i\,\mathbf{A}\mathbf{v} = (\mu\,\mathbf{u} - \nu\,\mathbf{v}) + i(\nu\,\mathbf{u} + \mu\,\mathbf{v}).$$

Equating real and imaginary parts, remembering that $\mathbf{A}$ is real, gives

$$\mathbf{A}\mathbf{u} = \mu\,\mathbf{u} - \nu\,\mathbf{v},$$
$$\mathbf{A}\mathbf{v} = \nu\,\mathbf{u} + \mu\,\mathbf{v}.$$

Thus

$$\mathbf{A}[\mathbf{u} \quad \mathbf{v}] = [\mathbf{u} \quad \mathbf{v}]\begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}.$$

Solution 1.50

For the eigenvalue λ, the equation for the eigenvector is

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

having solutions $[b \quad \mu \pm i\nu - a]^T$, where

$$\lambda = \mu \pm i\nu = \frac{\operatorname{tr}\mathbf{A} \pm i\sqrt{|\Delta|}}{2}.$$

Thus the real part of either of these eigenvectors is $[b \quad \mu - a]^T = [b \quad \frac{1}{2}(d-a)]^T$ and the imaginary part is $[0 \quad \pm\,\nu]^T = [0 \quad \pm\frac{1}{2}\sqrt{|\Delta|}]^T$.

Solution 1.51

What must be avoided at all costs, if the new coordinate system is to be well defined, is that two distinct points should be assigned the same coordinates. If this were the case, then there would be two distinct column vectors of Cartesian coordinates, say $\mathbf{x}_1$ and $\mathbf{x}_2$, such that $\mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{x}_2$, or equivalently such that

$$\mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) = 0.$$

But then there would be a non-zero vector, namely $\mathbf{x}_1 - \mathbf{x}_2$, mapped to the zero vector by $\mathbf{A}$, and $\mathbf{A}$ would be singular. But provided that $\mathbf{A}$ is non-singular this possibility can never happen.

The column vector $\mathbf{A}[1 \quad 0]^T$ is just the first column of $\mathbf{A}$, and $\mathbf{A}[0 \quad 1]^T$ is the second column. Thus the columns of $\mathbf{A}$ give the oblique coordinates of the points whose Cartesian coordinates are $(1, 0)$ and $(0, 1)$.

Solution 1.52

Suppose that $\mathbf{u}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, necessarily non-zero because $\mathbf{A}$ is non-singular. Then $\mathbf{u}$ determines a line, whose Cartesian equation is $uy = vx$ if $\mathbf{u} = [u \quad v]^T$; the equation of this line is the same in oblique coordinates as it is in Cartesians (that is to say, its equation in oblique coordinates is $uy' = vx'$). For every point on this line, the coordinate transformation amounts simply to multiplying the Cartesian coordinates by the common factor λ, so that if (x, y) are the Cartesian coordinates of a point on this line, the oblique coordinates of the same point are $(\lambda x, \lambda y)$. Thus along this line the transformation uniformly dilates ($|\lambda| < 1$) or expands ($|\lambda| > 1$), together with a reflection if $\lambda < 0$.

Solution 1.53

$$x(u,v) = \sqrt{u}\cos v, \qquad y(u,v) = \sqrt{u}\sin v$$

give

$$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{u}}\cos v, \qquad \frac{\partial x}{\partial v} = -\sqrt{u}\sin v,$$
$$\frac{\partial y}{\partial u} = \frac{1}{2\sqrt{u}}\sin v, \qquad \frac{\partial y}{\partial v} = \sqrt{u}\cos v.$$

Thus

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2\sqrt{u}}\cos v & -\sqrt{u}\sin v \\ \dfrac{1}{2\sqrt{u}}\sin v & \sqrt{u}\cos v \end{vmatrix} = \frac{1}{2\sqrt{u}} \times \sqrt{u}(\cos^2 v + \sin^2 v) = \tfrac{1}{2}.$$

The element of area is thus $\frac{1}{2}\delta u\,\delta v$.

Solution 1.54

First, the coordinate transformation must be expressed in the form in which the Cartesian coordinates are given as functions of the 'curvilinear' coordinates (curvilinear in quotes, because in this case the coordinate lines remain lines). This is simple to do: one just inverts the defining equations, to give

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}'.$$

The Jacobian matrix of this coordinate transformation is just the matrix $\mathbf{A}^{-1}$, and the Jacobian determinant is therefore $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$, so that the element of area is

$$|(\det \mathbf{A})^{-1}|\delta x'\delta y'.$$

The determinant of a matrix of the form

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix},$$

which represents a rotation, is 1, and so rotations are area-preserving.

Solution 1.55

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}},$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2},$$

so that

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}}.$$

But

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r,$$

and so provided that (x,y) and (r,θ) are the Cartesian and polar coordinates of the same point,

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r} = \left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)^{-1}.$$

Solution 1.56

This is an application of the chain rule:

$$x(t) = x(u(t),v(t)), \qquad y(t) = y(u(t),v(t)),$$

and so

$$\dot{x} = \frac{\partial x}{\partial u}\dot{u} + \frac{\partial x}{\partial v}\dot{v}, \qquad \dot{y} = \frac{\partial y}{\partial u}\dot{u} + \frac{\partial y}{\partial v}\dot{v},$$

which may be combined into a single matrix equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix},$$

or

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{u}}.$$

The u-coordinate curve is the image in the (x, y)-plane of the curve $(u + t, v)$, that is v = constant, in the (u, v)-plane. This latter curve is just a straight line through the point (u, v) parallel to the u-axis, and its tangent vector is $\mathbf{i}$ (a Cartesian unit vector in the (u, v)-plane). Thus

$$\mathbf{e}_u = \mathbf{J}\mathbf{i}.$$

Likewise,

$$\mathbf{e}_v = \mathbf{J}\mathbf{j}.$$

Now for a curve in the (u, v)-plane, say $(u(t), v(t))$,

$$\dot{\mathbf{u}} = \dot{u}\,\mathbf{i} + \dot{v}\,\mathbf{j},$$

and therefore

$$\dot{\mathbf{x}} = \mathbf{J}(\dot{u}\,\mathbf{i} + \dot{v}\,\mathbf{j}) = \dot{u}\,\mathbf{e}_u + \dot{v}\,\mathbf{e}_v.$$

Thus the tangent vector to the image curve in the (x, y)-plane has the same components relative to the tangent vectors to the u- and v-coordinate curves as the tangent vector to the original curve in the (u, v)-plane has relative to the Cartesian unit vectors there. (The vectors $\mathbf{e}_u, \mathbf{e}_v$ need not be unit vectors.)

Solution 1.57

The map

$$x(u, v) = \sqrt{2u}\cos v, \qquad y(u, v) = \sqrt{2u}\sin v$$

preserves area.

Solution 1.58

If $\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t)$ and $\dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t)$, then

$$\begin{aligned} \frac{d}{dt}\left(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\right) &= c_1\dot{\mathbf{x}}_1(t) + c_2\dot{\mathbf{x}}_2(t) \\ &= c_1\mathbf{A}\mathbf{x}_1(t) + c_2\mathbf{A}\mathbf{x}_2(t) \\ &= \mathbf{A}\left(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\right). \end{aligned}$$

Solution 1.59

If

$$\mathbf{B} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

then

$$\mathbf{B}\mathbf{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{bmatrix},$$

and so the equation $\dot{\mathbf{y}} = \mathbf{B}\mathbf{y}$, when written in component form, becomes

$$\dot{y}_1 = \lambda_1 y_1, \qquad \dot{y}_2 = \lambda_2 y_2.$$

The general solution of these equations is

$$y_1 = C_1 e^{\lambda_1 t}, \qquad y_2 = C_2 e^{\lambda_2 t},$$

where C_1 and C_2 are constants. This may be written in vector form as

$$\mathbf{y} = C_1 e^{\lambda_1 t}\,\mathbf{i} + C_2 e^{\lambda_2 t}\,\mathbf{j}.$$

The corresponding general solution of the original equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x} = \mathbf{P}\left(C_1 e^{\lambda_1 t}\,\mathbf{i} + C_2 e^{\lambda_2 t}\,\mathbf{j}\right) = C_1 e^{\lambda_1 t}\,\mathbf{P}\mathbf{i} + C_2 e^{\lambda_2 t}\,\mathbf{P}\mathbf{j}.$$

But $\mathbf{Pi}$ and $\mathbf{Pj}$ are the columns of $\mathbf{P}$, which are eigenvectors of $\mathbf{A}$, and so this solution is of the same general form as that produced by the trial method.

Solution 1.60

If

$$\mathbf{B} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

then the component form of the equation $\dot{\mathbf{y}} = \mathbf{B}\mathbf{y}$ is

$$\begin{aligned} \dot{y}_1 &= \lambda y_1, \\ \dot{y}_2 &= y_1 + \lambda y_2. \end{aligned}$$

These equations can be solved in a straightforward manner by solving the first and substituting the solution in the second. The solution to the first equation is

$$y_1 = C_1 e^{\lambda t}.$$

The second equation becomes, after substitution,

$$\dot{y}_2 = \lambda y_2 + C_1 e^{\lambda t},$$

which is similar to the equation solved in Example 1.8, and has the solution

$$y_2 = (C_1 t + C_2)\, e^{\lambda t}.$$

Solution 1.61

The equations for y_1 and y_2 are

$$\begin{aligned} \dot{y}_1 &= \nu y_2, \\ \dot{y}_2 &= -\nu y_1. \end{aligned}$$

On differentiating the first with respect to time and substituting in the second, these become

$$\ddot{y}_1 + \nu^2 y_1 = 0,$$

which has the general solution

$$y_1 = A\cos\nu t + B\sin\nu t,$$

where A and B are constants. Since $y_2 = \dot{y}_1/\nu$, we also have

$$y_2 = -A\sin\nu t + B\cos\nu t.$$

The general solution can be written in the vector form

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} A\cos\nu t + B\sin\nu t \\ -A\sin\nu t + B\cos\nu t \end{bmatrix}.$$

It is interesting to note that $A = y_1(0)$ and $B = y_2(0)$, and that the solution may be written

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos\nu t & \sin\nu t \\ -\sin\nu t & \cos\nu t \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}.$$

It is seen from this that the solutions may be generated by applying a time-dependent rotation matrix to the initial coordinates.

Solution 1.62

(i) The equations are

$$\dot{x}_1 = ax_1 + bx_2,$$
$$\dot{x}_2 = cx_1 + dx_2.$$

Differentiating the first of these again gives

$$\begin{aligned}\ddot{x}_1 &= a\dot{x}_1 + b\dot{x}_2\\ &= a\dot{x}_1 + b(cx_1 + dx_2)\\ &= a\dot{x}_1 + bcx_1 + d(\dot{x}_1 - ax_1)\\ &= (a+d)\dot{x}_1 - (ad - bc)x_1.\end{aligned}$$

If the same operations are applied to the equation for $\dot{x}_2$, one obtains

$$\begin{aligned}\ddot{x}_2 &= c\dot{x}_1 + d\dot{x}_2\\ &= c(ax_1 + bx_2) + d\dot{x}_2\\ &= a(\dot{x}_2 - dx_2) + bcx_2 + d\dot{x}_2\\ &= (a+d)\dot{x}_2 - (ad - bc)x_2,\end{aligned}$$

so x_2 satisfies the same equation.

(ii) The equation of part (i) may be written

$$\ddot{x} - (\operatorname{tr}\mathbf{A})\dot{x} + (\det\mathbf{A})x = 0,$$

where x stands for either x_1 or x_2. The auxiliary equation for this second-order differential equation is

$$\lambda^2 - \lambda \operatorname{tr}\mathbf{A} + \det\mathbf{A} = 0,$$

which is just the characteristic equation for the matrix of coefficients **A**.

(iii) When **A** has distinct real eigenvalues λ_1 and λ_2, the auxiliary equation has these roots and the second-order differential equation has the two independent solutions

$$x(t) = e^{\lambda_1 t} \qquad \text{and} \qquad x(t) = e^{\lambda_2 t}.$$

Thus since x_1 and x_2 both satisfy this equation, they must be expressible in the form

$$x_1(t) = A_1e^{\lambda_1 t} + B_1e^{\lambda_2 t},$$
$$x_2(t) = A_2e^{\lambda_1 t} + B_2e^{\lambda_2 t}.$$

However, the coefficients A_1, B_1, A_2 and B_2 are not independent; one effect of the repeated differentiation is to remove the interdependence of x_1 and x_2, and this must be restored by substituting these expressions in the original first-order equations. When this is done, and the coefficients of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ separately equated to each other, one finds the following conditions on the coefficients:

$$\lambda_1 A_1 = aA_1 + bA_2,$$
$$\lambda_1 A_2 = cA_1 + dA_2$$

and

$$\lambda_2 B_1 = aB_1 + bB_2,$$
$$\lambda_2 B_2 = cB_1 + dB_2.$$

Thus $[A_1 \quad A_2]^T$ and $[B_1 \quad B_2]^T$ must be eigenvectors of **A** corresponding to the eigenvalues λ_1 and λ_2 respectively, and the original solutions of the system of equations are recovered.

(iv) The solutions of the auxiliary equation are $\mu \pm i\nu$, and the general solution of the second-order differential equation is

$$x(t) = e^{\mu t}(A\cos\nu t + B\sin\nu t).$$

Thus

$$x_1(t) = e^{\mu t}(A_1\cos\nu t + B_1\sin\nu t)$$

and

$$x_2(t) = e^{\mu t}(A_2\cos\nu t + B_2\sin\nu t);$$

but, as before, the coefficients A_1, B_1, A_2 and B_2 are not independent. Substituting these expressions in the original first-order equations, and equating the coefficients of $e^{\mu t}\cos\nu t$ and $e^{\mu t}\sin\nu t$ separately to each other, gives the following conditions on the coefficients:

$$A_2 = B_1, \qquad B_2 = -A_1.$$

The general solution of the system for the given matrix **A** may be written

$$\mathbf{x}(t) = Ae^{\mu t}\begin{bmatrix}\cos\nu t\\ -\sin\nu t\end{bmatrix} + Be^{\mu t}\begin{bmatrix}\sin\nu t\\ \cos\nu t\end{bmatrix},$$

where $A = A_2 = B_1$, $B = B_2 = -A_1$.

Notice, for interest's sake, that the solution may be written

$$\mathbf{x}(t) = \begin{bmatrix}e^{\mu t} & 0\\ 0 & e^{\mu t}\end{bmatrix}\begin{bmatrix}\cos\nu t & \sin\nu t\\ -\sin\nu t & \cos\nu t\end{bmatrix}\mathbf{x}(0).$$

Thus the solution may be generated by applying first a time-dependent rotation, then a time-dependent scaling transformation, to the vector of the initial conditions.

Solution 1.63

The velocity is given by

$$v(t) = \frac{F_0}{m}t + v_0.$$

(i) If v_0 and F_0 have the same sign, then $v(t)$ does not change sign for $t > 0$ and increases in magnitude with increasing t. Thus the particle continues to move away from its starting point, in the direction defined by the sign of v_0, with ever-increasing speed.

(ii) If v_0 and F_0 have opposite signs, then there is some positive value of t, say τ, for which $v(\tau) = 0$. The particle moves away from its starting point, in the direction defined by v_0, with decreasing speed, until $t = \tau$; at this instant its velocity is momentarily zero. The particle then reverses its direction of motion, and comes back towards its starting point with increasing speed. It reaches the starting point at time $t = 2\tau$, and thereafter continues in the same direction, away from the starting point, just as in part (i).

Solution 1.64

If $k > 0$, then the force term kx will be positive for positive x, negative for negative x; thus this component of the force will always be directed away from the origin. If $k < 0$, on the other hand, then the force term will be negative for positive x, positive for negative x, and this component of the force will always be directed towards the origin. The total force is zero at

$$x_0 = -\frac{F_0}{k},$$

and

$$x(t) = -\frac{F_0}{k}$$

is a constant solution of the equation of motion; thus the particle can remain at rest indefinitely at this point, which is therefore a point of equilibrium.

In the attractive case $k < 0$, so one may write $k = -\omega^2 m$; the equation of motion becomes

$$\ddot{x} + \omega^2(x - x_0) = 0,$$

with general solution

$$x(t) = x_0 + A\cos\omega t + B\sin\omega t, \qquad \text{where} \quad x_0 = -\frac{F_0}{k}.$$

The constant A is equal to $x(0) - x_0$, the initial position of the particle relative to the equilibrium position, while $B = \dot{x}(0)/\omega$.

In the repulsive case $k > 0$, so $k = \gamma^2 m$, and the general solution of the equation of motion

$$\ddot{x} - \gamma^2(x - x_0) = 0$$

is

$$\begin{aligned} x &= x_0 + C\cosh\gamma t + D\sinh\gamma t \\ &= x_0 + \tfrac{1}{2}(C + D)e^{\gamma t} + \tfrac{1}{2}(C - D)e^{-\gamma t}. \end{aligned}$$

Solution 1.65

The potential energy is

$$U(x) = -F_0 x - \tfrac{1}{2}kx^2.$$

Solution 1.66

The total mechanical energy, in terms of the momentum, is

$$E = \frac{p^2}{2m} + V(x).$$

Solution 1.67

For $\mathbf{F} = mg\,\mathbf{i}$,

$$\mathbf{F}\cdot\mathbf{e}_\theta = -mg\sin\theta,$$

and so

$$W = -\int d\theta\,(-mg\sin\theta) = -mg\cos\theta.$$

The equation of motion is

$$mR\ddot{\theta} = \mathbf{F}\cdot\mathbf{e}_\theta = -mg\sin\theta$$

or

$$\ddot{\theta} + \frac{g}{R}\sin\theta = 0.$$

(With the approximation $\sin\theta \simeq \theta$, this gives the simple harmonic motion equation, leading to periodic motion with period $2\pi\sqrt{R/g}$ for the small oscillations of a circular pendulum, for example. For the motion of the 'hot-wheels' car, this approximation is not valid; but the point of interest in this case is whether the car remains on the track. This it does so long as $T \geq 0$, that is,

$$mR\dot{\theta}^2 + \mathbf{F}\cdot\mathbf{e}_r = mR\dot{\theta}^2 + mg\cos\theta \geq 0.$$

But from the equation of motion for $\ddot{\theta}$ it follows (see the text immediately after Exercise 1.68, page 50) that

$$\tfrac{1}{2}mR\dot{\theta}^2 + W = \tfrac{1}{2}mR\dot{\theta}^2 - mg\cos\theta = \text{constant} \; = \frac{m}{2R}v_0^2 - mg,$$

where v_0 is the car's speed at the bottom of the track; thus the car remains on the track so long as

$$\cos\theta \geq \frac{2gR - v_0^2}{3gR}.$$

If $v_0^2 > 5gR$, then this condition is satisfied for all θ and so the car never leaves the track, but will 'loop the loop'.)

Solution 1.68

The function W is given by

$$W(\theta) = -\int_{\theta_0}^{\theta} d\phi\, F(\phi),$$

where $F = \mathbf{F} \cdot \mathbf{e}_\theta$ and θ_0 is the zero level for W, so that $W(\theta_0) = 0$. In order that W be periodic we must have

$$\begin{aligned} 0 = W(\theta + 2\pi) - W(\theta) &= -\left(\int_{\theta_0}^{\theta+2\pi} d\phi\, F(\phi) - \int_{\theta_0}^{\theta} d\phi\, F(\phi)\right) \\ &= -\int_{\theta}^{\theta+2\pi} d\phi\, F(\phi) \\ &= -\int_{0}^{2\pi} d\phi\, F(\phi), \end{aligned}$$

the last step being a consequence of the fact that F is periodic. Thus the condition for W to be periodic is that the integral of the force component $\mathbf{F} \cdot \mathbf{e}_\theta$ from 0 to 2π, that is over a period, must be zero.

Solution 1.69

The particle's total energy is

$$E = \tfrac{1}{2} m R^2 \dot{\theta}^2 + V = \tfrac{1}{2} I \dot{\theta}^2 + V = \frac{1}{2I} p^2 + V.$$

Solution 1.70

(i) Acceleration: $[\mathrm{LT}^{-2}]$.
(ii) Linear momentum: $[\mathrm{MLT}^{-1}]$.
(iii) Energy: $[\mathrm{ML}^2\mathrm{T}^{-2}]$.
(iv) Moment of inertia: $[\mathrm{ML}^2]$.
(v) Angular momentum: $[\mathrm{ML}^2\mathrm{T}^{-1}]$.
(vi) Angle, being definable as the ratio of two lengths, is a dimensionless quantity.
(vii) Angular speed: $[\mathrm{T}^{-1}]$.

Solution 1.71

Since angle is dimensionless, W and $dW/d\theta$ have the same dimensions; thus W has the dimensions of force, which are $[\mathrm{MLT}^{-2}]$, while the dimensions of energy are $[\mathrm{ML}^2\mathrm{T}^{-2}]$. But the extra length in RW gives this quantity the dimensions of energy.

Solution 1.72

The dimensions of k are $[\mathrm{L}^{-3}\mathrm{T}^2]$.

Solution 1.73

The dimensions of G are $[\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}]$, so those of $2GM/R$ are $[\mathrm{L}^2\mathrm{T}^{-2}]$. The given formula is clearly not correct on dimensional grounds; the correct formula is

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}}.$$

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